Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 210 (2007) 375–403
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Cohomological construction of relative twists

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Received 20 August 2005; accepted 7 July 2006

Available online 15 September 2006

Communicated by Pavel Etingof

Abstract

Let $\mathfrak{g}$ be a complex, semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and D a subdiagram of the Dynkin diagram of $\mathfrak{g}$. Let $\mathfrak{g}_D \subset \mathfrak{l}_D \subseteq \mathfrak{g}$ be the corresponding semi-simple and Levi subalgebras and consider two invariant solutions $\Phi \in (U\mathfrak{g}^{\otimes 3}[[\hbar]])^{\mathfrak{g}}$ and $\Phi_D \in (U\mathfrak{g}_D^{\otimes 3}[[\hbar]])^{\mathfrak{g}_D}$ of the pentagon equation for $\mathfrak{g}$ and $\mathfrak{g}_D$ respectively. Motivated by the theory of quasi-Coxeter quasitriangular quasibialgebras [V. Toledano Laredo, Quasi-Coxeter algebras, Dynkin diagram cohomology and quantum Weyl groups, math.QA/0506529], we study in this paper the existence of a *relative twist*, that is an element $F \in (U\mathfrak{g}^{\otimes 2}[[\hbar]])^{\mathfrak{l}_D}$ such that the twist of Φ by F is Φ_D. Adapting the method of Donin and Shnider [J. Donin, S. Shnider, Cohomological construction of quantized universal enveloping algebras, Trans. Amer. Math. Soc. 349 (1997) 1611–1632], who treated the case of an empty D, so that $\mathfrak{l}_D = \mathfrak{h}$ and $\Phi_D = 1^{\otimes 3}$, we give a cohomological construction of such an F under the assumption that Φ_D is the image of Φ under the generalised Harish-Chandra homomorphism $(U\mathfrak{g}^{\otimes 3})^{\mathfrak{l}_D} \to (U\mathfrak{g}_D^{\otimes 3})^{\mathfrak{g}_D}$. We also show that F is unique up to a gauge transformation if $\mathfrak{l}_D$ is of corank 1 or F satisfies $F^{\Theta} = F^{21}$ where $\Theta \in \mathrm{Aut}(\mathfrak{g})$ is an involution acting as -1 on $\mathfrak{h}$.

Keywords: Quasibialgebras; Associators; Relative twist equation; r-matrix cohomology

1. Introduction

Let $\mathfrak{g}$ be a complex, semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $R_{\mathfrak{g}} = \{\alpha\} \subset \mathfrak{h}^*$ the corresponding root system and $D_{\mathfrak{g}}$ the Dynkin diagram of $\mathfrak{g}$ relative to a choice $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ of simple roots of $\mathfrak{g}$. Let $D \subseteq D_{\mathfrak{g}}$ be a subdiagram of the Dynkin diagram of $\mathfrak{g}$ and denote by

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2006.07.019

$$\mathfrak{g}_D \subseteq \mathfrak{l}_D \subseteq \mathfrak{g}$$

the corresponding diagrammatic subalgebra, i.e., the semi-simple subalgebra generated by the root vectors corresponding to the simple roots in D, and Levi subalgebra $\mathfrak{l}_D = \mathfrak{g}_D + \mathfrak{h}$ respectively. Note that

$$\mathfrak{l}_D = \mathfrak{g}_D \oplus \mathfrak{c}_D$$

where the centre $\mathfrak{c}_D$ of $\mathfrak{l}_D$ is spanned by the fundamental coweights $\lambda_j^\vee \in \mathfrak{h}$, with j such that $\alpha_j \notin D$.

Let $\hbar$ be a formal variable and consider two fixed, invariant elements

$$\Phi \in 1 + \hbar^2 \big(U\mathfrak{g}^{\otimes 3}[[\hbar]]\big)^{\mathfrak{g}} \quad \text{and} \quad \Phi_D \in 1 + \hbar^2 \big(U\mathfrak{g}_D^{\otimes 3}[[\hbar]]\big)^{\mathfrak{g}_D}$$

satisfying the pentagon equation

$$\mathrm{id}^{\otimes 2} \otimes \Delta(\Psi) \cdot \Delta \otimes \mathrm{id}^{\otimes 2}(\Psi) = 1 \otimes \Psi \cdot \mathrm{id} \otimes \Delta \otimes \mathrm{id}(\Psi) \cdot \Psi \otimes 1. \tag{1.1}$$

We shall be concerned in this paper with the cohomological solution of the following *relative twist equation*

$$(\Phi)_F := 1 \otimes F \cdot \mathrm{id} \otimes \Delta(F) \cdot \Phi \cdot \Delta \otimes \mathrm{id}\big(F^{-1}\big) \cdot F^{-1} \otimes 1 = \Phi_D \tag{1.2}$$

with respect to an element F which is invariant under the adjoint action of $\mathfrak{l}_D$

$$F \in 1 + \hbar \big(U\mathfrak{g}^{\otimes 2}[[\hbar]]\big)^{\mathfrak{l}_D}.$$

Our motivation for studying (1.2) comes from the theory of quasi-Coxeter quasitriangular quasibialgebras [10]. These are, informally speaking, bialgebras which carry representations of both Artin's braid groups B_n and the generalised braid group $B_{\mathfrak{g}}$ of type $\mathfrak{g}$ on the tensor products of their finite-dimensional modules. One of the main results in [10] is the rigidity of quasi-Coxeter quasitriangular quasibialgebra structures on $U\mathfrak{g}[[\hbar]]$. In conjunction with the results of [11], this shows in particular that the monodromy of the Casimir connection introduced in [7] is described by Lusztig's quantum Weyl group operators [6], thus proving a conjecture formulated independently by the author [8,9] and De Concini (unpublished). The rigidity result of [10] depends on Drinfeld's uniqueness theorem for quasitriangular quasibialgebra deformations of $U\mathfrak{g}$ [4] and on the uniqueness, up to gauge transformations, of solutions of (1.2) when $\mathfrak{l}_D$ is of corank 1.

Rather than incorporating the required uniqueness result into [10], we decided to study the existence of solutions of (1.2) as well and present our results in a separate publication. These may, in fact, be of independent interest since the relevant deformation complex turns out to be a perturbation of the Chevalley–Eilenberg complex for a suitable, non-coboundary Lie algebra structure on $\mathfrak{g}^*$. Our method is very close to that of Donin–Shnider [2, §3] who solved Eq. (1.2) when D is empty, so that $\mathfrak{l}_D = \mathfrak{h}$ and $\Phi_D = 1^{\otimes 3}$, and Φ satisfies in addition

$$\Phi^{321} = \Phi^{-1} \quad \text{and} \quad \Phi^{\Theta} = \Phi \tag{1.3}$$

where $\Theta \in \mathrm{Aut}(\mathfrak{g})$ is an involution acting as -1 on $\mathfrak{h}$. The possibility of laddering down, that is solving (1.2) only when $|D_{\mathfrak{g}} \setminus D| = 1$ allows us to bypass the use of (1.3) and to construct in

Section 5 a suitable F under the sole assumption that Φ_D is the projection of Φ with respect to the generalised Harish-Chandra homomorphism $(U\mathfrak{g}^{\otimes 3})^{\mathfrak{l}_D} \to (U\mathfrak{g}_D^{\otimes 3})^{\mathfrak{g}_D}$ defined in Section 2. Our proof proceeds along the lines of Donin and Shnider's, the main difference being in the cohomology theory needed to deal with secondary obstructions, which is defined and computed in Section 4. The uniqueness of solutions of (1.2) is obtained in Section 6 under the weaker assumption that the infinitesimal of Φ projects onto that of Φ_D and that either $\mathfrak{l}_D$ is of corank 1 or F satisfies $F^{\Theta} = F^{21}$. Section 3 contains some standard material on the classical Yang–Baxter equations.

Remark 1.1. A non-cohomological proof of the existence of F may be given in the case where Φ and Φ_D are Lie associators by adapting Etingof and Kazhdan's method [5]. The latter corresponds to the case when $\mathfrak{l}_D = \mathfrak{h}$ but can be modified by replacing the Verma modules used in [5] by their generalised counterparts obtained by inducing from the parabolic subalgebra $\mathfrak{p}_D \subset \mathfrak{g}$ corresponding to D.

2. Generalised Harish-Chandra homomorphisms

For each $k \geqslant 1$, we define below an algebra homomorphism

$$\pi_D^k : \left(U\mathfrak{g}^{\otimes k}\right)^{\mathfrak{c}_D} \to U\mathfrak{l}_D^{\otimes k}$$

which restricts to the identity on $U\mathfrak{l}_D^{\otimes k}$ and is equivariant with respect to adjoint action of $\mathfrak{l}_D$. For $D = \emptyset$ and $k = 1$, π_D^k is the Harish-Chandra homomorphism $\pi : U\mathfrak{g}^{\mathfrak{h}} \to U\mathfrak{h}$. The definition of π_D^k is similar to that of π, see, e.g., [1, §7.4.1–7.4.3] which we follow closely. Write

$$\mathfrak{g} = \mathfrak{n}_D^- \oplus \mathfrak{l}_D \oplus \mathfrak{n}_D^+$$

where the nilpotent subalgebras $\mathfrak{n}_D^{\pm}$ are spanned by the roots vectors e_α, f_α respectively, with α ranging over the positive roots of $\mathfrak{g}$ not lying in the root system R_D of $\mathfrak{g}_D$. Set

$$I_k = \left(U\mathfrak{g}^{\otimes k}\right)^{\mathfrak{c}_D} \cap \sum_{i=1}^{k} U\mathfrak{g}^{\otimes k} \cdot \left(\mathfrak{n}_D^+\right)_i$$

where, for $y \in U\mathfrak{g}$,

$$y_i = 1^{\otimes(i-1)} \otimes y \otimes 1^{\otimes(k-i)} \in U\mathfrak{g}^{\otimes k}.$$

Proposition 2.1.

(i) $I_k = (U\mathfrak{g}^{\otimes k})^{\mathfrak{c}_D} \cap \sum_{i=1}^k (\mathfrak{n}_D^-)_i \cdot U\mathfrak{g}^{\otimes k}$.
(ii) *I_k is a two-sided ideal in $(U\mathfrak{g}^{\otimes k})^{\mathfrak{c}_D}$ invariant under the adjoint action of $\mathfrak{l}_D$.*
(iii) $(U\mathfrak{g}^{\otimes k})^{\mathfrak{c}_D} = I_k \oplus U\mathfrak{l}_D^{\otimes k}$.

Proof. (i) By the PBW theorem,

$$U\mathfrak{g}^{\otimes k} \cong U\mathfrak{n}_D^{-\otimes k} \otimes U\mathfrak{l}_D^{\otimes k} \otimes U\mathfrak{n}_D^{+\otimes k}$$

is spanned by the monomials

$$u\big(q_i^j;x;p_i^j\big)=f_{\beta_1,1}^{q_1^1}\cdots f_{\beta_m,1}^{q_m^1}\cdots f_{\beta_1,k}^{q_1^k}\cdots f_{\beta_m,k}^{q_m^k}\cdot x\cdot e_{\beta_1,1}^{q_1^1}\cdots e_{\beta_m,1}^{q_m^1}\cdots e_{\beta_1,k}^{q_1^k}\cdots e_{\beta_m,k}^{q_m^k}$$

where $x\in U\mathfrak{l}_D^{\otimes k}$, $\beta_1,\dots,\beta_m$ are the positive roots in $R_{\mathfrak{g}}\setminus R_D$ and $q_i^j,p_i^j\in\mathbb{N}$. Let $\iota^*:\mathfrak{h}^*\to\mathfrak{c}_D^*$ be the restriction map. Since $u(q_i^j;x;p_i^j)$ has weight $\iota^*\sum_{i,j}p_{i,j}\beta_j-\iota^*\sum_{i,j}q_{i,j}\beta_j$ for the adjoint action of $\mathfrak{c}_D$, $(U\mathfrak{g}^{\otimes k})^{\mathfrak{c}_D}$ is spanned by the $u(q_i^j;x;p_i^j)$ such that

$$\iota^*\sum_{i,j}p_{i,j}\beta_j=\iota^*\sum_{i,j}q_{i,j}\beta_j.$$

Note that, since each β_j restricts on $\mathfrak{c}_D$ to a non-trivial linear combination of the simple roots $\alpha_\ell\notin D$, with non-negative coefficients, $\iota^*\sum_{i,j}p_{i,j}\beta_j=0$ iff $\sum_{i,j}p_{i,j}=0$. It follows that

$$\begin{aligned}I_k&=\big\langle u\big(q_i^j;x;p_i^j\big)\in\big(U\mathfrak{g}^{\otimes k}\big)^{\mathfrak{c}_D}\big\rangle_{\sum_{i,j}p_i^j>0}\\&=\big\langle u\big(q_i^j;x;p_i^j\big)\in\big(U\mathfrak{g}^{\otimes k}\big)^{\mathfrak{c}_D}\big\rangle_{\sum_{i,j}q_i^j>0}\\&=\big(U\mathfrak{g}^{\otimes k}\big)^{\mathfrak{c}_D}\cap\sum_{i=1}^k\big(\mathfrak{n}_D^-\big)_i\cdot U\mathfrak{g}^{\otimes k}\end{aligned}$$

as claimed. (ii) I_k is a left ideal by definition and, by (i), it is also a right ideal. It is moreover invariant under the adjoint action of $\mathfrak{l}_D$ since $\mathfrak{n}_D^\pm$ are. (iii) is now obvious. □

Corollary 2.2. *The projection π_D^k of $(U\mathfrak{g}^{\otimes k})^{\mathfrak{c}_D}$ onto $U\mathfrak{l}_D^{\otimes k}$ defined by the ideal I_k is equivariant for the adjoint action of $\mathfrak{l}_D$ and therefore gives rise to the following commutative diagram of algebra homomorphisms*

$$\begin{array}{ccccc}\big(U\mathfrak{g}^{\otimes k}\big)^{\mathfrak{c}_D}&\xrightarrow{\pi_D^k}&U\mathfrak{l}_D^{\otimes k}&\longrightarrow&U\mathfrak{g}_D^{\otimes k}\\ \cup&&\cup&&\cup\\ \big(U\mathfrak{g}^{\otimes k}\big)^{\mathfrak{l}_D}&\xrightarrow{\pi_D^k}&\big(U\mathfrak{l}_D^{\otimes k}\big)^{\mathfrak{l}_D}&\longrightarrow&\big(U\mathfrak{g}_D^{\otimes k}\big)^{\mathfrak{g}_D}\end{array}$$

where the rightmost horizontal arrows are induced by the Lie algebra projection $\mathfrak{l}_D\to\mathfrak{g}_D$.

Definition 2.3. We denote the composition of the horizontal arrows by $\overline{\pi}_D^k$ and refer to $\overline{\pi}_D^k$ or π_D^k as generalised Harish-Chandra homomorphisms.

Note that π_D^k and $\overline{\pi}_D^k$ are equivariant under the natural action of the symmetric group $\mathfrak{S}_k$. We record for later use the following two results:

Proposition 2.4. *For any $i,l\leqslant k$, $x\in(U\mathfrak{g}^{\otimes k})^{\mathfrak{c}_D}$ and $y\in(U\mathfrak{g}^{\otimes l})^{\mathfrak{c}_D}$, one has*

$$\mathrm{id}^{\otimes i}\otimes\Delta\otimes\mathrm{id}^{\otimes(k-i-1)}\circ\pi_D^k(x)=\pi_D^{k+1}\circ\mathrm{id}^{\otimes i}\otimes\Delta\otimes\mathrm{id}^{\otimes(k-i-1)}(x),\tag{2.1}$$

$$1^{\otimes i}\otimes\pi_D^l(y)\otimes1^{\otimes(k-l-i)}=\pi_D^k\big(1^{\otimes i}\otimes y\otimes1^{\otimes(k-i-l)}\big).\tag{2.2}$$

These identities remain valid if π^k_D, π^{k+1}_D and π^l_D are replaced by $\overline{\pi}^k_D$, $\overline{\pi}^{k+1}_D$ and $\overline{\pi}^l_D$, respectively.

Proof. Let

$$\gamma = \mathrm{id}^{\otimes i} \otimes \Delta \otimes \mathrm{id}^{\otimes (k-i-1)} : U\mathfrak{g}^{\otimes k} \to U\mathfrak{g}^{\otimes (k+1)}.$$

Since γ is equivariant for the adjoint action of $\mathfrak{g}$, it maps $(U\mathfrak{g}^{\otimes k})^{\mathfrak{c}_D}$ to $(U\mathfrak{g}^{\otimes (k+1)})^{\mathfrak{c}_D}$ so that the right-hand side of (2.1) is well defined. One readily checks that

$$\gamma(I_k) \subset I_{k+1} \quad \text{and that} \quad \gamma\big(U\mathfrak{l}_D^{\otimes k}\big) \subset U\mathfrak{l}_D^{\otimes (k+1)}$$

so that (2.1) holds. (2.2) is proved in the same way. The fact that these identities hold when π^j_D is replaced by $\overline{\pi}^j_D$ throughout follows from the fact that $\overline{\pi}^j_D = \pi^{\otimes j} \circ \pi^j_D$ where $\pi : U\mathfrak{l}_D \to U\mathfrak{g}_D$ is a Hopf algebra homomorphism. □

Corollary 2.5. *Let $d_H : U\mathfrak{g}^{\otimes k} \to U\mathfrak{g}^{\otimes (k+1)}$ be the Hochschild differential given by*

$$d_H x = 1 \otimes x + \sum_{i=1}^{k} (-1)^i \mathrm{id}^{\otimes (i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes (k-i)}(x) + (-1)^{k+1} x \otimes 1. \tag{2.3}$$

Then,

$$d_H \circ \pi^k_D = \pi^{k+1}_D \circ d_H \quad \text{and} \quad d_H \circ \overline{\pi}^k_D = \overline{\pi}^{k+1}_D \circ d_H.$$

3. Classical Yang–Baxter equations

We review below some well-known results on the classical Yang–Baxter equations due to Drinfeld [3].

Define the classical Yang–Baxter map $\mathrm{YB} : \mathfrak{g}^{\otimes 2} \otimes \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 3}$ by

$$\mathrm{YB}(r,s) = \big[r^{12}, s^{13} + s^{23}\big] + \big[r^{13}, s^{23}\big] + \big[s^{12}, r^{13} + r^{23}\big] + \big[s^{13}, r^{23}\big].$$

Identify the exterior algebra $\bigwedge \mathfrak{g}$ with its image in the tensor algebra $T\mathfrak{g}$ via the antisymmetrisation map

$$X_1 \wedge \cdots \wedge X_k \longrightarrow \mathrm{Alt}_k(X_1 \otimes \cdots \otimes X_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma(1)} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}.$$

One readily checks that if $r, s \in \bigwedge^2 \mathfrak{g}$, then

$$\mathrm{YB}(r,s) = 6\,\mathrm{Alt}_3\big[r^{12}, s^{13}\big]. \tag{3.1}$$

Lemma 3.1. *If $r = r_1 \wedge r_2$, $s = s_1 \wedge s_2 \in \bigwedge^2 \mathfrak{g}$, then*

$$\mathrm{YB}(r,s) = \frac{3}{2} \sum_{1 \leqslant i,j \leqslant 2} [r_i, s_j] \wedge r_{3-i} \wedge s_{3-j}. \tag{3.2}$$

Proof. We have

$$4\left[r^{12}, s^{13}\right] = [r_1, s_1] \otimes r_2 \otimes s_2 - [r_1, s_2] \otimes r_2 \otimes s_1 - [r_2, s_1] \otimes r_1 \otimes s_2 + [r_2, s_2] \otimes r_1 \otimes s_1.$$

Antisymmetrising both sides and using (3.1), we find (3.2). □

Let $(\cdot,\cdot)$ be a non-degenerate, ad-invariant, symmetric bilinear form on $\mathfrak{g}$ and let $\Omega = \sum_i x_i \otimes x^i$, where $\{x_i\}, \{x^i\}$ are dual basis of $\mathfrak{g}$ with respect to $(\cdot,\cdot)$, be the corresponding symmetric, invariant tensor in $\mathfrak{g} \otimes \mathfrak{g}$. It is well known that $[\Omega_{12}, \Omega_{23}]$ lies in $(\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}$ and generates it if $\mathfrak{g}$ is simple. Let

$$r_{\mathfrak{g}} = \sum_{\alpha \succ 0} \frac{(\alpha,\alpha)}{2} \cdot e_\alpha \wedge f_\alpha \in \bigwedge^2 \mathfrak{g} \tag{3.3}$$

where $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ are root vectors such that $[e_\alpha, f_\alpha] = h_\alpha$ so that

$$(e_\alpha, f_\alpha) = \frac{1}{2}\big([h_\alpha, e_\alpha], f_\alpha\big) = \frac{1}{2}\big(h_\alpha, [e_\alpha, f_\alpha]\big) = \frac{1}{2}(h_\alpha, h_\alpha) = \frac{2}{(\alpha,\alpha)}. \tag{3.4}$$

By the following result, $r_{\mathfrak{g}}$ is a solution of the modified classical Yang–Baxter equation (MCYBE), that is the equation

$$\left[r_{\mathfrak{g}}^{12}, r_{\mathfrak{g}}^{23} + r_{\mathfrak{g}}^{13}\right] + \left[r_{\mathfrak{g}}^{13}, r_{\mathfrak{g}}^{23}\right] \in \left(\bigwedge^3 \mathfrak{g}\right)^{\mathfrak{g}}. \tag{3.5}$$

Proposition 3.2 *(Drinfeld).*

$$\mathrm{YB}(r_{\mathfrak{g}}, r_{\mathfrak{g}}) = \frac{1}{2}[\Omega_{12}, \Omega_{23}].$$

Remark 3.3. We shall refer to $r_{\mathfrak{g}}$ given by (3.3) as the standard (Drinfeld) solution of the MCYBE corresponding to the bilinear form $(\cdot,\cdot)$.

4. Classical r-matrices and Chevalley–Eilenberg cohomology

4.1. The aim of this section is to compute the cohomology of the complex

$$\left(\left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}_D}, d\right) \quad \text{where } d = [\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]$$

is given by the Schouten bracket with the difference of the standard solutions of the modified classical Yang–Baxter equations for $\mathfrak{g}$ and $\mathfrak{g}_D$ respectively.

The computation is carried out by identifying d with a perturbation of the Chevalley–Eilenberg differential on $\bigwedge \mathfrak{g} = \bigwedge(\mathfrak{g}^*)^*$ induced by a suitable Lie algebra structure on $\mathfrak{g}^*$. When

$D = \emptyset$, so that $\mathfrak{l}_D = \mathfrak{h}$, this identification is well known and follows readily from the fact that the relevant Lie algebra structure on $\mathfrak{g}^*$ is given in terms of the cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ defined by

$$\delta(X) = [\![r_{\mathfrak{g}}, X]\!] = -\operatorname{ad}(X) r_{\mathfrak{g}}.$$

When $D \neq \emptyset$ the relevant Lie algebra structure on $\mathfrak{g}^*$ is described in Section 4.4 and is not of coboundary type. We begin with a few reminders.

4.2. Recall that the Schouten bracket

$$[\![\cdot,\cdot]\!] : \bigwedge^{k} \mathfrak{g} \otimes \bigwedge^{l} \mathfrak{g} \to \bigwedge^{k+l-1} \mathfrak{g}$$

on the exterior algebra $\bigwedge \mathfrak{g}$ is defined by

$$\begin{aligned} &[\![X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l]\!] \\ &\quad = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \widehat{Y_j} \wedge \cdots \wedge Y_l. \end{aligned} \tag{4.1}$$

The Schouten bracket satisfies

$$[\![\underline{X}, \underline{Y}]\!] = -(-1)^{(k-1)(l-1)} [\![\underline{Y}, \underline{X}]\!]$$

for any $\underline{X} \in \bigwedge^k \mathfrak{g}$ and $\underline{Y} \in \bigwedge^l \mathfrak{g}$ and

$$\big[\!\big[\underline{X}, [\![\underline{Y}, \underline{Z}]\!] \big]\!\big] = \big[\!\big[[\![\underline{X}, \underline{Y}]\!], \underline{Z} \big]\!\big] + (-1)^{(k-1)(l-1)} \big[\!\big[\underline{Y}, [\![\underline{X}, \underline{Z}]\!] \big]\!\big]$$

for any such $\underline{X}, \underline{Y}$ and $\underline{Z} \in \bigwedge \mathfrak{g}$, and therefore endows $\bigwedge \mathfrak{g}$ with the structure of a $\mathbb{Z}$-graded Lie algebra, provided its grading is defined by

$$\deg\Big(\bigwedge^{k} \mathfrak{g} \Big) = k - 1.$$

Moreover, since

$$[\![\underline{X}, \underline{Y} \wedge \underline{Z}]\!] = [\![\underline{X}, \underline{Y}]\!] \wedge \underline{Z} + (-1)^{(k-1)l} \underline{Y} \wedge [\![\underline{X}, \underline{Z}]\!]$$

for any $\underline{X} \in \bigwedge^k \mathfrak{g}$ and $\underline{Y} \in \bigwedge^l \mathfrak{g}$, the map $\underline{X} \to [\![\underline{X}, \cdot]\!]$ is a homomorphism of $\bigwedge \mathfrak{g}$ into the $\mathbb{Z}$-graded Lie algebra of derivations of the exterior algebra $\bigwedge \mathfrak{g}$ endowed with its standard grading.

Note that any $r \in \bigwedge^2 \mathfrak{g}$ defines a degree 1 derivation $d_r = [\![r, \cdot]\!]$ of $\bigwedge \mathfrak{g}$. Its square is readily computed from

$$d_r^2(\underline{Y}) = \big[\!\big[r, [\![r, \underline{Y}]\!] \big]\!\big] = \big[\!\big[[\![r, r]\!], \underline{Y} \big]\!\big] - \big[\!\big[r, [\![r, \underline{Y}]\!] \big]\!\big] = \big[\!\big[[\![r, r]\!], \underline{Y} \big]\!\big] - d_r^2(\underline{Y}).$$

Since d_r^2 is also an algebra derivation of $\bigwedge \mathfrak{g}$, and $[\![\underline{X}, Y]\!] = -\operatorname{ad}(Y)\underline{X}$ for any $\underline{X} \in \bigwedge \mathfrak{g}$ and $Y \in \mathfrak{g}$, d_r is a differential if, and only if

$$[\![r, r]\!] \in \left(\bigwedge^3 \mathfrak{g}\right)^{\mathfrak{g}}$$

and therefore, by Lemma 3.1, iff r is a solution of the MCYBE (3.5).

4.3. Let now

$$r_{\mathfrak{g}} \in \bigwedge^2 \mathfrak{g} \quad \text{and} \quad r_{\mathfrak{g}_D} \in \bigwedge^2 \mathfrak{g}_D$$

be solutions of the MCYBE for $\mathfrak{g}$ and $\mathfrak{g}_D$ respectively such that $r_{\mathfrak{g}} - r_{\mathfrak{g}_D}$ is invariant under $\mathfrak{g}_D$. This is the case for example if both $r_{\mathfrak{g}}$ and $r_{\mathfrak{g}_D}$ are the standard solutions (3.3) of MCYBE relative to a non-degenerate, ad-invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ and its restriction to $\mathfrak{g}_D$ respectively. Indeed, $\mathfrak{n}_D^+$ and $\mathfrak{n}_D^-$ are invariant under the adjoint action of $\mathfrak{g}_D$ and $(\cdot,\cdot)$ yields a $\mathfrak{g}_D$-equivariant identification $(\mathfrak{n}_D^+)^* \cong \mathfrak{n}_D^-$ with respect to which

$$r_{\mathfrak{g}} - r_{\mathfrak{g}_D} = \sum_{\alpha \in R_{\mathfrak{g}}^+ \setminus R_D} \frac{(\alpha, \alpha)}{2} \cdot e_\alpha \wedge f_\alpha \tag{4.2}$$

is the image in $\bigwedge^2(\mathfrak{n}_D^+ \oplus \mathfrak{n}_D^-)$ of

$$\mathrm{id}_{\mathfrak{n}_D^+} \in \operatorname{End}\big(\mathfrak{n}_D^+\big) \cong \mathfrak{n}_D^+ \otimes \mathfrak{n}_D^- \subset \big(\mathfrak{n}_D^+ \oplus \mathfrak{n}_D^-\big)^{\otimes 2}$$

under the projection $(\mathfrak{n}_D^+ \oplus \mathfrak{n}_D^-)^{\otimes 2} \to \bigwedge^2(\mathfrak{n}_D^+ \oplus \mathfrak{n}_D^-)$. We shall need the following simple

Lemma 4.1. *For any $\underline{X} = X_1 \wedge \cdots \wedge X_k \in \bigwedge^k \mathfrak{g}$, the following holds on $\bigwedge \mathfrak{g}$*

$$[\![\underline{X}, \cdot]\!] = (-1)^{k-1} \cdot \sum_{i=1}^{k} (-1)^{i-1} e(X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge X_k) \cdot \operatorname{ad}(X_i) \tag{4.3}$$

where $e(\underline{Y})$ is exterior multiplication by $\underline{Y}$. In particular, if $\underline{X} \in \bigwedge \mathfrak{g}_D$ and $\underline{Y} \in (\bigwedge \mathfrak{g})^{\mathfrak{g}_D}$, then $[\![\underline{X}, \underline{Y}]\!] = 0$.

Proposition 4.2. *Let $r_{\mathfrak{g}}$, $r_{\mathfrak{g}_D}$ be solutions of the MCYBE for $\mathfrak{g}$, $\mathfrak{g}_D$ respectively such that $r_{\mathfrak{g}} - r_{\mathfrak{g}_D}$ is invariant under $\mathfrak{g}_D$. Then*:

(i) $[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]$ *leaves $(\bigwedge \mathfrak{g})^{\mathfrak{g}_D}$ invariant.*
(ii) *Its restriction to $(\bigwedge \mathfrak{g})^{\mathfrak{g}_D}$ coincides with that of $[\![r_{\mathfrak{g}}, \cdot]\!]$ and is therefore a differential.*
(iii) $[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, r_{\mathfrak{g}} - r_{\mathfrak{g}_D}]\!] = [\![r_{\mathfrak{g}}, r_{\mathfrak{g}}]\!] - [\![r_{\mathfrak{g}_D}, r_{\mathfrak{g}_D}]\!]$.

Proof. (i) Since $r_{\mathfrak{g}} - r_{\mathfrak{g}_D}$ is invariant under $\mathfrak{g}_D$, and the Schouten bracket is equivariant for the adjoint action of $\mathfrak{g}$, $[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]$ leaves $(\bigwedge \mathfrak{g})^{\mathfrak{g}_D}$ invariant. (ii) By Lemma 4.1, $[\![r_{\mathfrak{g}_D}, \underline{Y}]\!] = 0$ for any $\underline{Y} \in (\bigwedge \mathfrak{g})^{\mathfrak{g}_D}$ so that

$$[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \underline{Y}]\!] = [\![r_{\mathfrak{g}}, \underline{Y}]\!] \tag{4.4}$$

for any such $\underline{Y}$. (iii) Since $r_{\mathfrak{g}} - r_{\mathfrak{g}_D}$ is invariant under $\mathfrak{g}_D$, we find, by (4.4)

$$\begin{aligned}[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, r_{\mathfrak{g}} - r_{\mathfrak{g}_D}]\!] &= [\![r_{\mathfrak{g}}, r_{\mathfrak{g}} - r_{\mathfrak{g}_D}]\!] = [\![r_{\mathfrak{g}}, r_{\mathfrak{g}}]\!] - [\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, r_{\mathfrak{g}_D}]\!] - [\![r_{\mathfrak{g}_D}, r_{\mathfrak{g}_D}]\!] \\ &= [\![r_{\mathfrak{g}}, r_{\mathfrak{g}}]\!] - [\![r_{\mathfrak{g}_D}, r_{\mathfrak{g}_D}]\!]\end{aligned}$$

as claimed. □

Remark 4.3. Note that the proof of (iii) only uses the $\mathfrak{g}_D$-invariance of $r_{\mathfrak{g}} - r_{\mathfrak{g}_D}$. Thus, if $r_{\mathfrak{g}} \in \bigwedge^2 \mathfrak{g}$ is a solution of the MCYBE and $r_{\mathfrak{g}_D} \in \bigwedge^2 \mathfrak{g}_D$ is such that $r_{\mathfrak{g}} - r_{\mathfrak{g}_D}$ is invariant under $\mathfrak{g}_D$, then $r_{\mathfrak{g}_D}$ is a solution of the MCYBE for $\mathfrak{g}_D$. Moreover, if $r_{\mathfrak{g}}$ is the standard solution of the MCYBE then so is $r_{\mathfrak{g}_D}$. Indeed, if $\pi \in \mathrm{End}(\bigwedge^2 \mathfrak{g})$ is the projection onto $\mathfrak{g}_D$-invariants, then

$$r_{\mathfrak{g}} - r_{\mathfrak{g}_D} = \pi(r_{\mathfrak{g}} - r_{\mathfrak{g}_D}) = \pi\big(r_{\mathfrak{g}} - \bar{\pi}_D^2(r_{\mathfrak{g}}) + \bar{\pi}_D^2(r_{\mathfrak{g}}) - r_{\mathfrak{g}_D}\big) = r_{\mathfrak{g}} - \bar{\pi}_D^2(r_{\mathfrak{g}})$$

where the last equality follows from the $\mathfrak{g}_D$-invariance of $r_{\mathfrak{g}} - \bar{\pi}_D^2(r_{\mathfrak{g}})$ and the fact that $\pi(\bar{\pi}_D^2(r_{\mathfrak{g}}) - r_{\mathfrak{g}_D}) \in (\bigwedge^2 \mathfrak{g}_D)^{\mathfrak{g}_D} = \{0\}$. Thus, $r_{\mathfrak{g}_D} = \bar{\pi}_D^2(r_{\mathfrak{g}})$ is the standard solution of the MCYBE for $\mathfrak{g}_D$.

4.4. Identify $\mathfrak{g}^*$ and $\mathfrak{g}$ as vector spaces by using the bilinear form $(\cdot,\cdot)$, and endow $\mathfrak{g}^*$ with the following Lie algebra structure:

$$\mathfrak{g}^* = \big(\mathfrak{n}_D^+ \oplus \bar{\mathfrak{n}}_D^-\big) \rtimes (\mathfrak{g}_D \oplus \mathfrak{c}_D) \tag{4.5}$$

where $\bar{\mathfrak{n}}_D^-$ is $\mathfrak{n}_D^-$ with the opposite bracket, $\mathfrak{g}_D$ acts on $\mathfrak{n}_D^\pm$ by the adjoint action and $\mathfrak{c}_D$ acts on $\mathfrak{n}_D^\pm$ by $\pm 1/2$ times the adjoint action. Denoting the corresponding bracket on $\mathfrak{g}^*$ by $[\cdot,\cdot]^*$, we therefore have

$$\begin{aligned}[x, y]^* &= [x_D, y_D] + [x_D, y_+ + y_-] + [x_+ + x_-, y_D] \\ &\quad + [x_+, y_+] - [x_-, y_-] + \frac{1}{2}[x_0, y_+ - y_-] + \frac{1}{2}[x_+ - x_-, y_0]\end{aligned} \tag{4.6}$$

where, $z_D \in \mathfrak{g}_D$, $z_\pm \in \mathfrak{n}_D^\pm$ and $z_0 \in \mathfrak{c}_D$ are the components of $z \in \mathfrak{g}^*$ corresponding to the decomposition (4.5). Thus, $\mathfrak{g}_D$ is a Lie subalgebra of $\mathfrak{g}^*$ and its coadjoint action on $(\mathfrak{g}^*)^* = \mathfrak{g}$ coincides with its adjoint action on $\mathfrak{g}$.

Let now $\delta \in \mathrm{End}(\bigwedge \mathfrak{g})$ be the differential obtained by regarding $\bigwedge \mathfrak{g}$ as the Chevalley–Eilenberg complex of $\mathfrak{g}^*$. The following result identifies $[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]$ with a perturbation of δ.

Theorem 4.4. *If $r_{\mathfrak{g}}$ and $r_{\mathfrak{g}_D}$ are the standard solutions of the MCYBE corresponding to $(\cdot,\cdot)$ and its restriction to $\mathfrak{g}_D$ respectively, the following holds on $\bigwedge \mathfrak{g}$:*

$$[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!] = 2\delta + e(v_i) \cdot \big[\mathrm{ad}(v_i) \cdot (1 + 2P_+)\big]^\wedge \tag{4.7}$$

where $\{v_i\}$, $\{v^i\}$ *are basis of* $\mathfrak{g}_D$ *dual with respect to* $(\cdot,\cdot)$, $P_+ : \mathfrak{g} \to \mathfrak{n}_D^+$ *is the projection corresponding to the decomposition* (4.5) *and* $T \in \mathfrak{gl}(\mathfrak{g}) \to T^\wedge \in \mathfrak{gl}(\bigwedge \mathfrak{g})$ *is the Lie algebra homomorphism given by*

$$T^\wedge X_1 \wedge \cdots \wedge X_k = \sum_{i=1}^{k} X_1 \wedge \cdots \wedge T X_i \wedge \cdots \wedge X_k.$$

Proof. It is sufficient to check (4.7) on elements of $\mathfrak{g} \subset \bigwedge \mathfrak{g}$ since both sides are degree 1 algebra derivations of $\bigwedge \mathfrak{g}$. In turn, it is easier to check that the transposes of both sides coincide as maps $\bigwedge^2 \mathfrak{g} \to \mathfrak{g}$. By definition, $\delta^t = [\cdot,\cdot]^*$. Since $e(v)^t = \iota(v)$ where $\iota(v)$ is the contraction operator defined by

$$\iota(v) Y_1 \wedge \cdots \wedge Y_l = \sum_{i=1}^{l} (-1)^{i-1} (v, Y_i) Y_1 \wedge \cdots \wedge \widehat{Y_i} \wedge \cdots \wedge Y_l$$

and $\operatorname{ad}(X)^t = -\operatorname{ad}(X)$ for any $X \in \mathfrak{g}$, we find, using (4.2) and (4.3)

$$[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]^t = \sum_{\alpha \in R_{\mathfrak{g}}^+ \setminus R_D} \frac{(\alpha,\alpha)}{2} \big(\operatorname{ad}(e_\alpha) \iota(f_\alpha) - \operatorname{ad}(f_\alpha) \iota(e_\alpha) \big)$$

which, applied to $u \wedge v \in \bigwedge^2 \mathfrak{g}$ yields

$$\begin{aligned}
&\sum_{\alpha \in R_{\mathfrak{g}}^+ \setminus R_D} \frac{(\alpha,\alpha)}{2} \big((f_\alpha, u)[e_\alpha, v] - (f_\alpha, v)[e_\alpha, u] - (e_\alpha, u)[f_\alpha, v] + (e_\alpha, v)[f_\alpha, u] \big) \\
&\quad = [u_+, v] + [u, v_+] - [u_-, v] - [u, v_-] \\
&\quad = 2[u_+, v_+] - 2[u_-, v_-] + [u_0, v_+ - v_-] + [u_+ - u_-, v_0] \\
&\qquad + [u_+, v_D] + [u_D, v_+] - [u_-, v_D] - [u_D, v_-].
\end{aligned}$$

Comparing with (4.6), we see that this is equal to

$$\begin{aligned}
&2[u, v]^* - 2[u_D, v_D] - [u_D, v_+] - [u_+, v_D] - 3[u_D, v_-] - 3[u_-, v_D] \\
&\quad = 2[u, v]^* - [u_D, v] - [u, v_D] - 2[u_D, v_-] - 2[u_-, v_D] \\
&\quad = 2[u, v]^* - (1 + 2P_-)\big([u_D, v] + [u, v_D]\big)
\end{aligned}$$

where P_- is the projection onto $\mathfrak{n}_D^-$ which commutes with the adjoint action of $\mathfrak{g}_D$. Noting that, for $x, y \in \mathfrak{g}$, one has

$$\operatorname{ad}\big(v^i\big) \iota(v_i) x \wedge y = (v_i, x)\big[v^i, y\big] - (v_i, y)\big[v^i, x\big] = [x_D, y] + [x, y_D]$$

we therefore find

$$[\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]^t = 2[\cdot,\cdot]^* - (1 + 2P_-) \operatorname{ad}(v_i) \iota\big(v^i\big)$$

which yields (4.7) since $P_-^t = P_+$. $\square$

4.5. Note that $(\bigwedge\mathfrak{g})^{\mathfrak{l}_D}$ is a subcomplex of $((\bigwedge\mathfrak{g})^{\mathfrak{g}_D}, [\![r_\mathfrak{g}-r_{\mathfrak{g}_D},\cdot]\!])$ since $r_\mathfrak{g}-r_{\mathfrak{g}_D}$ is of weight zero. Note also that the restriction of $[\![r_\mathfrak{g}-r_{\mathfrak{g}_D},\cdot]\!]$ to

$$\Big(\bigwedge\mathfrak{l}_D\Big)^{\mathfrak{l}_D}=\Big(\bigwedge\mathfrak{g}_D\Big)^{\mathfrak{g}_D}\widehat{\otimes}\bigwedge\mathfrak{c}_D\subset\Big(\bigwedge\mathfrak{g}\Big)^{\mathfrak{l}_D}$$

is zero since $r_\mathfrak{g}-r_{\mathfrak{g}_D}$ is invariant under $\mathfrak{l}_D$.

Theorem 4.5. *If $r_\mathfrak{g}$, $r_{\mathfrak{g}_D}$ are the standard solutions of the MCYBE for $\mathfrak{g}$, $\mathfrak{g}_D$ respectively, the inclusions*

$$\Big(\Big(\bigwedge\mathfrak{l}_D\Big)^{\mathfrak{l}_D},0\Big)\longrightarrow\Big(\Big(\bigwedge\mathfrak{g}\Big)^{\mathfrak{l}_D},[\![r_\mathfrak{g}-r_{\mathfrak{g}_D},\cdot]\!]\Big)\to\Big(\Big(\bigwedge\mathfrak{g}\Big)^{\mathfrak{g}_D},[\![r_\mathfrak{g}-r_{\mathfrak{g}_D},\cdot]\!]\Big)$$

are quasi-isomorphisms.

Proof. Denote $[\![r_\mathfrak{g}-r_{\mathfrak{g}_D},\cdot]\!]$ by d. It is sufficient to find an $\mathfrak{l}_D$-equivariant, diagonalisable operator $C\in\operatorname{End}(\bigwedge\mathfrak{g})$ with kernel $\bigwedge\mathfrak{l}_D$ and an $\mathfrak{l}_D$-equivariant homotopy $h\in\operatorname{End}(\bigwedge\mathfrak{g})$ such $dh+hd=C$. Noting that $\mathfrak{c}_D\subset\mathfrak{g}^*$ acts on $\bigwedge\mathfrak{g}$ via the coadjoint action with non-negative weights only so that the corresponding subspace of invariants is precisely $\bigwedge\mathfrak{l}_D$, we see that a suitable C is given by the Casimir operator

$$C=\operatorname{ad}^*(t_i)\operatorname{ad}^*\big(t^i\big)$$

where ad^* is the coadjoint action of $\mathfrak{g}^*$ on $\bigwedge\mathfrak{g}$ and $\{t_i\},\{t^i\}$ are dual basis of $\mathfrak{c}_D$ with respect to $(\cdot,\cdot)$. We claim that

$$h=\operatorname{ad}^*(t_i)\iota\big(t^i\big)$$

satisfies $dh+hd=2C$. It is well known that h satisfies $\delta h+h\delta=C$, where $\delta\in\operatorname{End}(\bigwedge\mathfrak{g})$ is the Chevalley–Eilenberg differential. Indeed,

$$\begin{aligned}\delta h+h\delta&=\delta\operatorname{ad}^*(t_i)\iota\big(t^i\big)+\operatorname{ad}^*(t_i)\iota\big(t^i\big)\delta\\&=\operatorname{ad}^*(t_i)\big(\delta\iota\big(t^i\big)+\iota\big(t^i\big)\delta\big)\\&=\operatorname{ad}^*(t_i)\operatorname{ad}^*\big(t^i\big)\end{aligned}$$

where we have used the fact that δ is equivariant for ad^* and the identity $\delta\iota(X)+\iota(X)\delta=\operatorname{ad}^*(X)$, $X\in\mathfrak{g}^*$. By Theorem 4.4, it therefore suffices to show that h anticommutes with

$$k=e(v_j)\cdot\big[(1+2P_+)\cdot\operatorname{ad}\big(v^j\big)\big]^\wedge.$$

Bearing in mind the following identities for $X,Y\in\mathfrak{g}$ and $T\in\mathfrak{gl}(\mathfrak{g})$:

$$\iota(X)e(Y)+e(Y)\iota(X)=(X,Y),\quad \big[T^\wedge,\iota(X)\big]=\iota(TX)\quad\text{and}\quad\big[T^\wedge,e(Y)\big]=e(TY),$$

and the fact that $\mathrm{ad}^*(X)^\wedge = \mathrm{ad}^*(X)$ for any $X \in \mathfrak{g}^*$, we find

$$\begin{aligned} kh &= e(v_j) \cdot \left[(1+2P_+) \cdot \mathrm{ad}\left(v^j\right)\right]^\wedge \cdot \mathrm{ad}^*(t_i)\iota\left(t^i\right) \\ &= \mathrm{ad}^*(t_i)e(v_j) \cdot \left[(1+2P_+) \cdot \mathrm{ad}\left(v^j\right)\right]^\wedge \cdot \iota\left(t^i\right) \\ &= -\,\mathrm{ad}^*(t_i)\iota\left(t^i\right)e(v_j) \cdot \left[(1+2P_+) \cdot \mathrm{ad}\left(v^j\right)\right]^\wedge \end{aligned}$$

as claimed. □

Since $(\bigwedge^i \mathfrak{g}_D)^{\mathfrak{g}_D} = 0$ for $i = 1, 2$, we obtain in particular the following

Corollary 4.6.

$$\begin{aligned} H^1\left(\left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}_D}; [\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]\right) &\cong \mathfrak{c}_D, \\ H^2\left(\left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}_D}; [\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]\right) &\cong \bigwedge^2 \mathfrak{c}_D, \\ H^3\left(\left(\bigwedge \mathfrak{g}\right)^{\mathfrak{g}_D}; [\![r_{\mathfrak{g}} - r_{\mathfrak{g}_D}, \cdot]\!]\right) &\cong \bigwedge^3 \mathfrak{c}_D \oplus \left(\bigwedge^3 \mathfrak{g}_D\right)^{\mathfrak{g}_D}. \end{aligned}$$

5. Existence of twists

5.1. Let

$$\Phi \in 1^{\otimes 3} + \hbar^2\left(U\mathfrak{g}^{\otimes 3}[\![\hbar]\!]\right)^{\mathfrak{g}}$$

be a solution of the pentagon equation (1.1). We shall need to assume that Φ is non-degenerate in the sense defined below. Write

$$\Phi = 1^{\otimes 3} + \hbar^2\varphi \mod \hbar^3 \quad \text{where } \varphi \in \left(U\mathfrak{g}^{\otimes 3}\right)^{\mathfrak{g}}.$$

Taking the coefficient of $\hbar^2$ in the pentagon relation for Φ, we find that $d_H\varphi = 0$ where d_H is the Hochschild differential given by (2.3). Thus,

$$\mathrm{Alt}_3(\varphi) \in \left(\bigwedge^3 \mathfrak{g}\right)^{\mathfrak{g}} = \bigoplus_i \left(\bigwedge^3 \mathfrak{g}_i\right)^{\mathfrak{g}_i} \tag{5.1}$$

where $\mathfrak{g}_i$ are the simple factors of $\mathfrak{g}$.

Definition 5.1. Φ is a non-degenerate solution of the pentagon equation if the components of $\mathrm{Alt}_3(\varphi)$ along the decomposition (5.1) are all non-zero.

Since each $(\bigwedge^3 \mathfrak{g}_i)^{\mathfrak{g}_i}$ is one-dimensional and generated by $[\Omega^i_{12}, \Omega^i_{23}]$, where $\Omega^i \in (\mathfrak{g}_i \otimes \mathfrak{g}_i)^{\mathfrak{g}_i}$ is the symmetric element corresponding to the Killing form of $\mathfrak{g}_i$, Φ is a non-degenerate solution of the pentagon equation iff

$$\mathrm{Alt}_3(\varphi) = \frac{1}{6}[\Omega_{12}, \Omega_{23}] \tag{5.2}$$

where $\Omega \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{g}}$ corresponds to a non-degenerate, ad-invariant, symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$.

Let now $D \subseteq D_{\mathfrak{g}}$ be a subdiagram, and set

$$\Phi_D = \overline{\pi}^3_D(\Phi) \in 1 + \hbar^2 \big(U\mathfrak{g}_D^{\otimes 3}[[\hbar]]\big)^{\mathfrak{g}_D}$$

where $\overline{\pi}^3_D$ is the generalised Harish-Chandra homomorphism defined in Section 2. By Proposition 2.4, Φ_D satisfies the pentagon equation. Note that Φ_D is non-degenerate if Φ is.

Theorem 5.2. *If Φ is non-degenerate, there exists an element*

$$F \in 1^{\otimes 2} + \hbar \big(U\mathfrak{g}^{\otimes 2}[[\hbar]]\big)^{\mathfrak{l}_D}$$

such that

$$(\Phi)_F = \Phi_D \quad \textit{and} \quad \overline{\pi}^2_D(F) = 1 \otimes 1. \tag{5.3}$$

Modulo $\hbar^2$, one has

$$F = 1^{\otimes 2} + \hbar(r_{\mathfrak{g}} - r_{\mathfrak{g}_D})$$

where $r_{\mathfrak{g}}$, $r_{\mathfrak{g}_D}$ are the standard solutions of the MCYBE for $\mathfrak{g}$ and $\mathfrak{g}_D$ corresponding to $(\cdot,\cdot)$. If Φ satisfies in addition

$$\Phi^{321} = \Phi^{-1} \quad \textit{and} \quad \Phi^{\Theta} = \Phi \tag{5.4}$$

where $\Theta \in \mathrm{Aut}(\mathfrak{g})$ is an involution acting as -1 on $\mathfrak{h}$, then F may be chosen such that

$$F^{\Theta} = F^{21}. \tag{5.5}$$

Remark 5.3. If Φ is an associator, that is satisfies in addition $\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}(\Phi) = 1^{\otimes 2}$, where $\varepsilon : U\mathfrak{g} \to \mathbb{C}$ is the counit, then so is Φ_D since one checks that $\mathrm{id}^{\otimes i} \otimes \varepsilon \otimes \mathrm{id}^{\otimes (k-i-1)} \circ \pi^k_D = \pi^{k-1}_D \circ \mathrm{id}^{\otimes i} \otimes \varepsilon \otimes \mathrm{id}^{\otimes (k-i-1)}$. In this case, it follows from (5.3) that F is a twist, i.e., satisfies $\varepsilon \otimes \mathrm{id}(F) = 1 = \mathrm{id} \otimes \varepsilon(F)$.

5.2. The proof of Theorem 5.2 is given in Sections 5.3–5.8. It closely follows the argument of Donin–Shnider [2, §3] where Theorem 5.2 is proved, under the additional assumption (5.4), in the case $D=\emptyset$. The reader familiar with Donin and Shnider's argument will readily recognize that a relevant difference is that the cohomology group

$$H^3\Big(\bigwedge\mathfrak{g};[\![r_\mathfrak{g},\cdot]\!]\Big)\cong\bigwedge^3\mathfrak{h} \tag{5.6}$$

which governs the secondary obstructions theory in [2] is replaced by the group

$$H^3\Big(\Big(\bigwedge\mathfrak{g}\Big)^{\mathfrak{g}_D};[\![r_\mathfrak{g}-r_{\mathfrak{g}_D},\cdot]\!]\Big)\cong\bigwedge^3\mathfrak{c}_D\oplus\Big(\bigwedge^3\mathfrak{g}_D\Big)^{\mathfrak{g}_D} \tag{5.7}$$

which was computed in Section 4. Another significant difference is that the possibility of laddering down from $D_\mathfrak{g}$ to D through intermediate diagrams, as explained in Section 5.3, allows in effect to assume that $\mathfrak{c}_D$ is at most two-dimensional, thus killing the first component of the secondary obstruction in (5.7) and rendering the assumption (5.4) unnecessary to prove the existence of F.

5.3. Although we will only use this from Section 5.7 onwards, note that we may assume that $|D_\mathfrak{g}\setminus D|\leqslant 2$. Indeed, assume Theorem 5.2 proved in this case and let

$$D_\mathfrak{g}=D_1\supset D_2\supset\cdots\supset D_{m-1}\supset D_m=D$$

be a nested chain of diagrams such that $|D_j\setminus D_{j+1}|\leqslant 2$. For any pair $D''\subseteq D'\subseteq D_\mathfrak{g}$, denote by $\mathfrak{c}_{D'',D'}\subset\mathfrak{h}$ the span of the fundamental coweights $\lambda_k^\vee$, with k such that $\alpha_k\in D'\setminus D''$ and by

$$\overline{\pi}^k_{D'',D'}:\big(U\mathfrak{g}_{D'}^{\otimes k}\big)^{\mathfrak{c}_{D'',D'}}\to U\mathfrak{g}_{D''}^{\otimes k}$$

the corresponding generalised Harish-Chandra homomorphism. Set $\Phi_1=\Phi$ and, for $j=1,\dots,m-1$

$$\Phi_{j+1}=\overline{\pi}^3_{D_{j+1},D_j}(\Phi_j)=\overline{\pi}^3_{D_{j+1},D_\mathfrak{g}}(\Phi)$$

so that $\Phi_m=\Phi_D$. Let

$$F_j\in 1^{\otimes 2}+\hbar\big(U\mathfrak{g}_{D_j}^{\otimes 2}[\![\hbar]\!]\big)^{\mathfrak{g}_{D_{j+1}}}$$

be such that

$$(\Phi_j)_{F_j}=\Phi_{j+1}\quad\text{and}\quad\overline{\pi}^2_{D_{j+1},D_j}(F_j)=1^{\otimes 2}$$

then

$$F=F_{m-1}\cdots F_1$$

is readily seen to satisfy (5.3). Note that if Φ satisfies in addition (5.4) then so does each Φ_j since $\overline{\pi}^k_D$ is equivariant for the action of the symmetric group $\mathfrak{S}_k$ and of Θ. In that case, choosing each F_j such that $F_j^\Theta=F_j^{21}$ yields an F which satisfies $F^\Theta=F^{21}$.

5.4. We begin by solving Eq. (5.3) mod $\hbar^2$. Let $f \in (U\mathfrak{g}^{\otimes 2})^{\mathfrak{l}_D}$ and set $F = 1^{\otimes 2} + \hbar f$. Since Φ and Φ_D are equal to $1^{\otimes 3}$ mod $\hbar^2$, the coefficient of $\hbar$ in $(\Phi)_F - \Phi_D$ is

$$1 \otimes f + \mathrm{id} \otimes \Delta(f) - \Delta \otimes \mathrm{id}(f) - f \otimes 1 = d_H f.$$

Thus, F is a solution of (5.3) mod $\hbar^2$ if, and only if, f is a Hochschild 2-cocycle such that $\overline{\pi}^2_D f = 0$.

5.5. Let now $n \geqslant 1$ and let

$$F = 1^{\otimes 2} + \hbar f + \cdots + \hbar^n f_n \in 1^{\otimes 2} + \hbar \bigl(U\mathfrak{g}^{\otimes 2}[[\hbar]]\bigr)^{\mathfrak{l}_D}$$

be a solution of (5.3) mod $\hbar^{n+1}$. We shall derive below a necessary and sufficient condition for (5.3) to possess a solution mod $\hbar^{n+2}$ of the form $\widetilde{F} = F + \hbar^{n+1} f_{n+1}$ where

$$f_{n+1} \in \bigl(U\mathfrak{g}^{\otimes 2}\bigr)^{\mathfrak{l}_D} \quad \text{satisfies} \quad \overline{\pi}^2_D(f_{n+1}) = 0.$$

Define $\xi \in U\mathfrak{g}^{\otimes 3}$ by

$$1 \otimes F \cdot \mathrm{id} \otimes \Delta(F) \cdot \Phi - \Phi_D \cdot F \otimes 1 \cdot \Delta \otimes \mathrm{id}(F) = \hbar^{n+1}\xi \quad \text{mod } \hbar^{n+2}. \tag{5.8}$$

Then, $\widetilde{F}$ is a solution of (5.3) mod $\hbar^{n+2}$ if, and only if $d_H f_{n+1} = -\xi$.

Lemma 5.4. *The element ξ is invariant under $\mathfrak{l}_D$ and satisfies*

$$d_H \xi = 0 \quad \text{and} \quad \overline{\pi}^3_D(\xi) = 0.$$

Proof. The invariance of ξ under $\mathfrak{l}_D$ follows from that of Φ, Φ_D and F. Since $F = 1^{\otimes 2}$ mod $\hbar$,

$$\hbar^{n+1}\xi = (\Phi)_F - \Phi_D \quad \text{mod } \hbar^{n+2}.$$

Since F is invariant under $\mathfrak{g}_D$, the restriction of

$$\Delta_F(\cdot) = F \cdot \Delta(\cdot) \cdot F^{-1}$$

to $U\mathfrak{g}_D$ is equal to Δ and Φ_D satisfies the pentagon equation with respect to Δ_F. Since this is also the case of $(\Phi)_F$, we find, working mod $\hbar^{n+2}$, that

$$0 = \mathrm{Pent}_{\Delta_F}\bigl((\Phi)_F\bigr) = \mathrm{Pent}_{\Delta_F}(\Phi_D) + \hbar^{n+1} d_H \xi = \hbar^{n+1} d_H \xi$$

where, for any $\Psi \in U\mathfrak{g}^{\otimes 3}$ and map $\widetilde{\Delta}: U\mathfrak{g} \to U\mathfrak{g}^{\otimes 2}$,

$$\mathrm{Pent}_{\widetilde{\Delta}}(\Psi) = 1 \otimes \Psi \cdot \mathrm{id} \otimes \widetilde{\Delta} \otimes \mathrm{id}(\Psi) \cdot \Psi \otimes 1 - \mathrm{id}^{\otimes 2} \otimes \widetilde{\Delta}(\Psi) \cdot \widetilde{\Delta} \otimes \mathrm{id}^{\otimes 2}(\Psi).$$

Finally, from $\overline{\pi}^2_D(F) = 1^{\otimes 2}$ and $\Phi_D = \overline{\pi}^3_D(\Phi)$, we get, using Proposition 2.4 that

$$\hbar^{n+1} \overline{\pi}^3_D \xi = \bigl(\overline{\pi}^3_D(\Phi)\bigr)_{\overline{\pi}^2_D(F)} - \Phi_D = 0. \qquad \square$$

Lemma 5.5. *If Φ and F satisfy* (5.4) *and* (5.5) *respectively, then*

$$\xi^{\Theta} = -\xi^{321}. \tag{5.9}$$

Proof. We have, working mod $\hbar^{n+2}$,

$$\begin{aligned}
\Phi_D^{\Theta} + \hbar^{n+1}\xi^{\Theta} &= 1 \otimes F^{21} \cdot \mathrm{id} \otimes \Delta\big(F^{21}\big) \cdot \Phi \cdot \Delta \otimes \mathrm{id}\big(\big(F^{21}\big)^{-1}\big) \otimes \big(F^{21}\big)^{-1} \otimes 1 \\
&= \big(F \otimes 1 \cdot \Delta \otimes \mathrm{id}(F) \cdot \Phi^{321} \cdot \mathrm{id} \otimes \Delta\big(F^{-1}\big) \cdot 1 \otimes F^{-1}\big)^{321} \\
&= \big(\big(1 \otimes F \cdot \mathrm{id} \otimes \Delta(F) \cdot \Phi \cdot \Delta \otimes \mathrm{id}\big(F^{-1}\big) \cdot F^{-1} \otimes 1\big)^{-1}\big)^{321} \\
&= \big(\big(\Phi_D + \hbar^{n+1}\xi\big)^{-1}\big)^{321} \\
&= \big(\Phi_D^{-1} - \hbar^{n+1}\xi\big)^{321}
\end{aligned}$$

whence (5.9) since $\Phi_D = \overline{\pi}_D^3(\Phi)$ satisfies (5.4). □

Corollary 5.6. *The element F may be extended to a solution of* (5.3) mod $\hbar^{n+2}$ *if, and only if* $\mathrm{Alt}_3\,\xi = 0$. *If in addition Φ, F satisfy* (5.4) *and* (5.5) *respectively, the extension may be chosen so as to satisfy* (5.5).

Proof. $\mathrm{Alt}_3\,\xi = 0$ if, and only if $\xi = d_H g$ for some $g \in U\mathfrak{g}^{\otimes 2}$, which may then be chosen invariant under $\mathfrak{l}_D$. By Corollary 2.5, we have

$$0 = \overline{\pi}_D^3 \xi = \overline{\pi}_D^3 d_H g = d_H \overline{\pi}_D^2 g$$

so that, setting $f_{n+1} = -(g - \overline{\pi}_D^2(g))$ we have

$$d_H f_{n+1} = -\xi \quad \text{and} \quad \overline{\pi}_D^2 f_{n+1} = 0$$

and $F + \hbar^{n+1} f_{n+1}$ is a solution of (5.3) mod $\hbar^{n+2}$. If Φ, F satisfy (5.4) and (5.5) respectively, then, by Lemma 5.5

$$d_H f_{n+1}^{\Theta} = -\xi^{\Theta} = \xi^{321} = -(d_H f_{n+1})^{321} = d_H f_{n+1}^{21}$$

so that $f'_{n+1} = 1/2(f_{n+1} + (f_{n+1}^{21})^{\Theta})$ satisfies

$$d_H f'_{n+1} = \xi, \quad \overline{\pi}_D^2 f'_{n+1} = 0 \quad \text{and} \quad \big(f'_{n+1}\big)^{\Theta} = \big(f'_{n+1}\big)^{21}$$

and $F + \hbar^{n+1} f'_{n+1}$ solves (5.3) mod $\hbar^{n+2}$ and satisfies (5.5). □

5.6. We consider first the case $n = 1$ so that $F = 1 + \hbar f$ where $f \in (U\mathfrak{g}^{\otimes 2})^{\mathfrak{l}_D}$ is a Hochschild 2-cocycle such that $\overline{\pi}_D^2(f) = 0$. By Lemma 5.9 below, adding a 2-coboundary to f does not affect the extendability of F to a solution mod $\hbar^3$. We may therefore assume that $f \in (\bigwedge^2 \mathfrak{g})^{\mathfrak{l}_D}$. In this case, since Φ and Φ_D are equal to $1^{\otimes 3}$ mod $\hbar^2$, we get

$$\xi = \varphi - \varphi_D + f^{23}\big(f^{12} + f^{13}\big) - f^{12}\big(f^{13} + f^{23}\big)$$

where $\Phi = 1 + \hbar^2\varphi \bmod \hbar^3$ and $\varphi_D = \overline{\pi}_D^3\varphi$. Thus, F extends to a solution mod $\hbar^3$ if, and only if,

$$\mathrm{Alt}_3(\varphi) - \mathrm{Alt}_3(\varphi_D) = \mathrm{Alt}_3\bigl(f^{12}\bigl(f^{13} + f^{23}\bigr) - f^{23}\bigl(f^{12} + f^{13}\bigr)\bigr).$$

We shall need the following

Lemma 5.7. *For any $f, \chi \in \bigwedge^2 \mathfrak{g}$, one has*

$$\begin{aligned}&\mathrm{Alt}_3\bigl(f^{12}\bigl(\chi^{13} + \chi^{23}\bigr) + \chi^{12}\bigl(f^{13} + f^{23}\bigr) - f^{23}\bigl(\chi^{12} + \chi^{13}\bigr) - \chi^{23}\bigl(f^{12} + f^{13}\bigr)\bigr)\\&\quad = [\![f, \chi]\!]\end{aligned} \tag{5.10}$$

where $[\![\cdot,\cdot]\!]$ is the Schouten bracket (4.1).

Proof. Since

$$\bigl(f^{12}\bigl(\chi^{13} + \chi^{23}\bigr)\bigr)^{(13)} = f^{23}\bigl(\chi^{13} + \chi^{12}\bigr),$$
$$\bigl(f^{12}\chi^{23}\bigr)^{(12)} = -f^{12}\chi^{13} \quad \text{and} \quad \bigl(\chi^{12}f^{13}\bigr)^{(23)} = \chi^{13}f^{12}$$

the left-hand side of (5.10) is equal to

$$\begin{aligned}&2\,\mathrm{Alt}_3\bigl(f^{12}\bigl(\chi^{13} + \chi^{23}\bigr) + \chi^{12}\bigl(f^{13} + f^{23}\bigr)\bigr)\\&\quad = 4\,\mathrm{Alt}_3\bigl(f^{12}\chi^{13} + \chi^{12}f^{13}\bigr) = 4\,\mathrm{Alt}_3\bigl(\bigl[f^{12}, \chi^{13}\bigr]\bigr) = \frac{2}{3}\,\mathrm{YB}(f, \chi) = [\![f, \chi]\!]\end{aligned}$$

where we used (3.1) and Lemma 3.1. □

Corollary 5.8. *Let $f \in (\bigwedge^2 \mathfrak{g})^{\mathfrak{l}_D}$ be such that $\overline{\pi}_D^2(f) = 0$. Then, the element*

$$F = 1 + \hbar f$$

extends to a solution $\widetilde{F}$ of (5.3) mod $\hbar^3$ *if, and only if,*

$$\frac{1}{2}[\![f, f]\!] = \mathrm{Alt}_3\,\varphi - \mathrm{Alt}_3\,\varphi_D.$$

In that case, and provided (5.4) *holds, $\widetilde{F}$ may be chosen so as to satisfy* (5.5).

Let $(\cdot,\cdot)$ be the non-degenerate, ad-invariant, symmetric bilinear form on $\mathfrak{g}$ such that

$$\mathrm{Alt}_3(\varphi) = \frac{1}{6}[\Omega_{12}, \Omega_{23}] \tag{5.11}$$

and $r_{\mathfrak{g}}, r_{\mathfrak{g}_D}$ be the standard solutions of the MCYBE determined by $(\cdot,\cdot)$ and its restriction to $\mathfrak{g}_D$ respectively, so that $\overline{\pi}_D^2(r_{\mathfrak{g}}) = r_{\mathfrak{g}_D}$. We henceforth set

$$f = r_{\mathfrak{g}} - r_{\mathfrak{g}_D} \in \left(\bigwedge^2 \mathfrak{g}\right)^{\mathfrak{l}_D}. \tag{5.12}$$

By Corollary 5.8, $F = 1 + \hbar f$ extends to a solution of (5.3) mod $\hbar^3$ which, in addition, satisfies (5.5) if (5.4) holds. Indeed, by Proposition 4.2, Lemma 3.1 and Proposition 3.2, we have

$$[\![f, f]\!] = [\![r_{\mathfrak{g}}, r_{\mathfrak{g}}]\!] - [\![r_{\mathfrak{g}_D}, r_{\mathfrak{g}_D}]\!] = \frac{2}{3}\big(\mathrm{YB}(r_{\mathfrak{g}}, r_{\mathfrak{g}}) - \mathrm{YB}(r_{\mathfrak{g}_D}, r_{\mathfrak{g}_D})\big)$$
$$= \frac{1}{3}\big([\Omega_{12}, \Omega_{23}] - \big[\Omega_{12}^D, \Omega_{23}^D\big]\big).$$

5.7. Assume now $n \geqslant 2$ and let

$$F = 1 + \hbar f + \hbar^2 f_2 + \cdots + \hbar^n f_n$$

be a solution of (5.3) mod $\hbar^{n+1}$. Let $\xi = \xi(f; f_2, \ldots, f_n) \in U\mathfrak{g}^{\otimes 3}$ be given by (5.8). By Section 5.5, ξ is a Hochschild 3-cocycle and F extends to a solution mod $\hbar^{n+2}$ if, and only if, ξ is a coboundary. This, however need not be the case. We note none-the-less that if, $\chi \in (U\mathfrak{g}^{\otimes 2})^{\mathfrak{l}_D}$ satisfies

$$d_H \chi = 0 \quad \text{and} \quad \overline{\pi}_D^2(\chi) = 0$$

then $F + \hbar^n \chi$ is also a solution of (5.3) mod $\hbar^{n+1}$ which could admit an extension mod $\hbar^{n+2}$. By the following result, the extendability of $F + \hbar^n \chi$ only depends upon the Hochschild cohomology class of χ.

Lemma 5.9. *If χ is a Hochschild 2-coboundary, then the element $F + \hbar^n \chi$ can be extended to a solution* mod $\hbar^{n+2}$ *of* (5.3) *if, and only if F can.*

Proof. It suffices to prove one implication since $F = (F + \hbar^n \chi) - \hbar^n \chi$. Write $\chi = d_H g$ with $g \in U\mathfrak{g}^{\mathfrak{l}_D}$. By Corollary 2.5,

$$0 = \overline{\pi}_D^2(\chi) = d_H \overline{\pi}_D^1 g$$

so that $\chi = d_H g'$ where $g' = (1 - \overline{\pi}_D^1)g$ is invariant under $\mathfrak{l}_D$ and lies in the kernel of $\overline{\pi}_D^1$. Let $\widetilde{F} = F + \hbar^{n+1} f_{n+1}$ be a solution mod $\hbar^{n+2}$ of (5.3). Then

$$\widetilde{F}' = u^{\otimes 2} \cdot \widetilde{F} \cdot \Delta(u)^{-1}$$

where $u = 1 + \hbar^n g$, is equal to $F + \hbar^n \chi$ mod $\hbar^{n+1}$ and solves (5.3) mod $\hbar^{n+2}$ since

$$\begin{aligned}
(\Phi)_{\widetilde{F}'} &= u^{\otimes 3} \cdot 1 \otimes \widetilde{F} \cdot \mathrm{id} \otimes \Delta(\widetilde{F}) \cdot \mathrm{id} \otimes \Delta\big(\Delta(u)\big)^{-1} \cdot \Phi \\
&\quad \cdot \Delta \otimes \mathrm{id}\big(\Delta(u)\big) \cdot \Delta \otimes \mathrm{id}(\widetilde{F}) \cdot \widetilde{F} \otimes 1 \cdot \big(u^{\otimes 3}\big)^{-1} \\
&= u^{\otimes 3} \cdot \Phi_D \cdot \big(u^{\otimes 3}\big)^{-1} \\
&= \Phi_D
\end{aligned}$$

where the first equality follows from the $\mathfrak{g}$-invariance of Φ and the last from the $\mathfrak{g}_D$-invariance of u. $\square$

We may therefore assume that χ lies in $(\bigwedge^2 \mathfrak{g})^{\mathfrak{l}_D}$. We then note that, for $n \geqslant 2$,

$$\xi(f; f_2, \dots, f_n + \chi) = \xi(f; f_2, \dots, f_n) + f^{23}(\chi^{12} + \chi^{13}) + \chi^{23}(f^{12} + f^{13}) \\ - f^{12}(\chi^{13} + \chi^{23}) - \chi^{12}(f^{13} + f^{23})$$

so that $F + \hbar^n \chi$ possesses an extension mod $\hbar^{n+2}$ if, and only if,

$$\mathrm{Alt}_3\big(\xi(f; f_2, \dots, f_n)\big) = [\![f, \chi]\!]$$

where we used Lemma 5.7.

Proposition 5.10. *The element* $\widetilde{\xi} = \mathrm{Alt}_3(\xi) \in (\bigwedge^3 \mathfrak{g})^{\mathfrak{l}_D}$ *satisfies*

$$[\![f, \widetilde{\xi}]\!] = 0 \quad \textit{and} \quad \overline{\pi}^2_D(\widetilde{\xi}) = 0.$$

We defer the proof of Proposition 5.10 to Section 5.8 in order to conclude the proof of Theorem 5.2. By Proposition 5.10, $\widetilde{\xi}$ is a 3-cocycle in $((\bigwedge \mathfrak{g})^{\mathfrak{l}_D}, [\![f, \cdot]\!])$ and we must show that it is a 3-coboundary. By Theorem 4.5

$$\widetilde{\xi} = [\![f, \chi]\!] + \eta \tag{5.13}$$

for some $\chi \in (\bigwedge^2 \mathfrak{g})^{\mathfrak{l}_D}$ and

$$\eta \in \left(\bigwedge^3 \mathfrak{l}_D\right)^{\mathfrak{l}_D} = \bigwedge^3 \mathfrak{c}_D \oplus \left(\bigwedge^3 \mathfrak{g}_D\right)^{\mathfrak{g}_D} = \left(\bigwedge^3 \mathfrak{g}_D\right)^{\mathfrak{g}_D}$$

where the first equality follows from the fact that $(\bigwedge^i \mathfrak{g}_D)^{\mathfrak{g}_D} = 0$ for $i = 1, 2$ and the second from the assumption that $|D_{\mathfrak{g}} \setminus D| \leqslant 2$ so that $\mathfrak{c}_D$ is at most two-dimensional. Applying $\overline{\pi}^3_D$ to both sides of (5.13), we find, since $\overline{\pi}^2_D(f) = 0$, that

$$0 = \overline{\pi}^3_D\big([\![f, \chi]\!] + \eta\big) = \eta.$$

Note that if Φ, F satisfy (5.4) and (5.5) respectively, then by Lemma 5.5

$$\widetilde{\xi}^{\Theta} = \mathrm{Alt}_3\big(\xi^{\Theta}\big) = \mathrm{Alt}_3\big(-\xi^{321}\big) = \widetilde{\xi}. \tag{5.14}$$

Since $f^{\Theta} = -f$, this implies

$$[\![f, \chi]\!] = [\![f, \chi]\!]^{\Theta} = -[\![f, \chi^{\Theta}]\!]$$

so that $\chi' = 1/2(\chi - \chi^{\Theta})$ satisfies $\chi'^{\Theta} = -\chi' = \chi'^{21}$ and $[\![f, \chi']\!] = \widetilde{\xi}$ and $F + \hbar^n \chi'$ is a solution of (5.3) mod $\hbar^{n+1}$ possessing an extension to a solution mod $\hbar^{n+2}$ satisfying (5.5). This completes the proof of Theorem 5.2. $\square$

Remark 5.11. If Φ satisfies (5.4), it is not necessary to ladder down, that is, assume that $|D_{\mathfrak{g}} \setminus D| \leqslant 2$. Indeed, by Theorem 4.5, there exist unique elements $u \in \bigwedge^3 \mathfrak{c}_D$, $v \in (\bigwedge^3 \mathfrak{g}_D)^{\mathfrak{g}_D}$ and a $\chi \in (\bigwedge^2 \mathfrak{g})^{\mathfrak{l}_D}$ such that

$$\widetilde{\xi} = u + v + [\![f, \chi]\!].$$

Since $\widetilde{\xi}$ and u are killed by $\overline{\pi}^3_D$, $v = 0$. Applying Θ to both sides and using (5.14), we find that $u^{\Theta} = u$. This however implies that $u = 0$ since Θ acts by -1 on $\bigwedge^3 \mathfrak{h} \supseteq \bigwedge^3 \mathfrak{c}_D$.

5.8. Proof of Proposition 5.10

We begin with some preliminary lemmas. Let $\widetilde{\Delta}: U\mathfrak{g} \to U\mathfrak{g}^{\otimes 2}$ be a linear map. Let

$$\xi = \xi_1 \otimes \cdots \otimes \xi_k \in U\mathfrak{g}^{\otimes k},$$

$i \leqslant k$, and write

$$\widetilde{\Delta}(\xi_i) = \sum_a \xi'_{i,a} \otimes \xi''_{i,a}.$$

For any enumeration $j_1, \ldots, j_{k+1}$ of $[1, k+1]$, we set

$$\xi^{j_1,\ldots,j_{i-1},j_i\, j_{i+1},j_{i+2},\ldots,j_{k+1}} = \sum_a \eta_a$$

where $\eta_a \in U\mathfrak{g}^{\otimes(k+1)}$ is the decomposable tensor with component ξ_ℓ in position j_ℓ if $\ell \leqslant i-1$ and in position $j_{\ell+1}$ if $\ell \geqslant i+1$ and components $\xi'_{i,a}, \xi''_{i,a}$ in positions j_i and j_{i+1} respectively. In other words,

$$\begin{aligned}\xi^{j_1,\ldots,j_{i-1},j_i\, j_{i+1},j_{i+2},\ldots,j_{k+1}} &= \sigma \xi^{1,\ldots,i-1,i\, i+1,i+2,\ldots,k+1} \\ &= \sigma \circ \mathrm{id}^{\otimes(i-1)} \otimes \widetilde{\Delta} \otimes \mathrm{id}^{\otimes(k-i)} \xi\end{aligned}$$

where $\sigma \in \mathfrak{S}_{k+1}$ is the permutation mapping ℓ to i_ℓ.

Lemma 5.12. *For any $\xi \in \bigwedge^k \mathfrak{g}$, one has*

$$\begin{aligned}&(k+1)\,\mathrm{Alt}_{k+1}\left(\sum_{i=1}^{k} (-1)^i \mathrm{id}^{\otimes(i-1)} \otimes \widetilde{\Delta} \otimes \mathrm{id}^{(k-i)} \xi\right) \\ &\quad = \sum_{1 \leqslant a < b \leqslant k+1} (-1)^{a+b} \big((\mathrm{Alt}_k \xi)^{ab,1,\ldots,\widehat{a},\ldots,\widehat{b},\ldots,k+1} - (\mathrm{Alt}_k \xi)^{ba,1,\ldots,\widehat{a},\ldots,\widehat{b},\ldots,k+1}\big).\end{aligned} \tag{5.15}$$

Proof. For any $i \in [1, k]$ and $a \neq b \in [1, k+1]$, set

$$\mathfrak{S}^{i:a,b}_{k+1} = \big\{\sigma \in \mathfrak{S}_{k+1} \mid \sigma(i) = a,\ \sigma(i+1) = b\big\}.$$

Then, the left-hand side of (5.15) is equal to

$$\frac{1}{k!}\sum_{\substack{1\leqslant a<b\leqslant k+1\\ 1\leqslant i\leqslant k\\ \sigma\in\mathfrak{S}_{k+1}^{i:a,b}}}(-1)^i(-1)^\sigma\bigl(\xi^{\sigma(1),\dots,\sigma(i-1),ab,\sigma(i+2),\dots,\sigma(k+1)}$$
$$-\,\xi^{\sigma(1),\dots,\sigma(i-1),ba,\sigma(i+2),\dots,\sigma(k+1)}\bigr)$$
$$=\frac{1}{k!}\sum_{\substack{1\leqslant a<b\leqslant k+1\\ 1\leqslant i\leqslant k\\ \sigma\in\mathfrak{S}_{k+1}^{i:a,b}}}(-1)^i(-1)^\sigma\bigl((\overline{\sigma}\xi)^{ab,1,\dots,\widehat{a},\dots,\widehat{b},\dots,k+1}-(\overline{\sigma}\xi)^{ba,1,\dots,\widehat{a},\dots,\widehat{b},\dots,k+1}\bigr)$$

where, for any $\sigma\in\mathfrak{S}_{k+1}^{i:a,b}$,

$$\overline{\sigma}\in\mathfrak{S}_k^i=\bigl\{\tau\in\mathfrak{S}_k \mid \tau(i)=1\bigr\}$$

is the permutation determined by the commutativity of the following diagram

$$\begin{array}{ccc}[1,k]\setminus\{i\} & \longrightarrow & [1,k+1]\setminus\{i,i+1\}\\ {\scriptstyle\overline{\sigma}}\big\downarrow & & \big\downarrow{\scriptstyle\sigma}\\ [1,k]\setminus\{1\} & \longrightarrow & [1,k+1]\setminus\{a,b\}\end{array}$$

where the horizontal arrow are the obvious monotone identifications. Noting that $\sigma\to\overline{\sigma}$ is an isomorphism of $\mathfrak{S}_{k+1}^{i:a,b}$ onto $\mathfrak{S}_k^i$ and deferring for the time being the proof that

$$(-1)^i(-1)^\sigma=(-1)^{a+b}(-1)^{\overline{\sigma}}\tag{5.16}$$

we see that the above is equal to

$$\frac{1}{k!}\sum_{\substack{1\leqslant a<b\leqslant k+1\\ 1\leqslant i\leqslant k\\ \overline{\sigma}\in\mathfrak{S}_k^i}}(-1)^{a+b}(-1)^{\overline{\sigma}}\bigl((\overline{\sigma}\xi)^{ab,1,\dots,\widehat{a},\dots,\widehat{b},\dots,k+1}-(\overline{\sigma}\xi)^{ba,1,\dots,\widehat{a},\dots,\widehat{b},\dots,k+1}\bigr)$$

and therefore to the right-hand side of (5.15). We turn now to the proof of (5.16). Let $\overline{\overline{\sigma}}\in\mathfrak{S}_{k-1}$ be the permutation determined by the commutativity of

$$\begin{array}{ccccc}[1,k]\setminus\{i\} & \longrightarrow & [1,k-1] & \longrightarrow & [1,k+1]\setminus\{i,i+1\}\\ {\scriptstyle\overline{\sigma}}\big\downarrow & & {\scriptstyle\overline{\overline{\sigma}}}\big\downarrow & & \big\downarrow{\scriptstyle\sigma}\\ [1,k]\setminus\{1\} & \longrightarrow & [1,k-1] & \longrightarrow & [1,k+1]\setminus\{a,b\}\end{array}$$

where the horizontal arrows are the obvious monotone identifications. Since $(-1)^{\overline{\sigma}}=(-1)^{\overline{\overline{\sigma}}}\cdot(-1)^{i-1}$, it suffices to prove that $(-1)^\sigma=(-1)^{\overline{\overline{\sigma}}}(-1)^{a+b-1}$. This clearly holds if $a=1$ and $b=2$. In the general case, letting $\tau\in\mathfrak{S}_{k+1}$ be the unique permutation such that τ is increasing on

$[1,k+1]\setminus\{a,b\}$, $\tau(a)=1$ and $\tau(b)=2$, so that $(-1)^\tau=(-1)^{a+b-1}$, and noting that $\overline{\overline{\tau\circ\sigma}}=\overline{\overline{\sigma}}$, we see that

$$(-1)^\sigma=(-1)^{a+b-1}(-1)^{\tau\circ\sigma}=(-1)^{a+b-1}(-1)^{\overline{\overline{\tau\circ\sigma}}}=(-1)^{a+b-1}(-1)^{\overline{\overline{\sigma}}}. \qquad \square$$

Lemma 5.13. *For any $Y, X_1,\dots,X_k\in\mathfrak{g}$, one has*

$$\begin{aligned} &Y\wedge X_1\wedge\cdots\wedge X_k\\ &\quad=\frac{1}{(k+1)!}\sum_{\substack{1\leqslant i\leqslant k+1\\ \tau\in\mathfrak{S}_k}}(-1)^{i-1}(-1)^\tau X_{\tau(1)}\otimes\cdots\otimes X_{\tau(i-1)}\otimes Y\otimes X_{\tau(i)}\otimes\cdots\otimes X_{\tau(k)}.\end{aligned} \tag{5.17}$$

Proof. Set $Z_1=Y$ and $Z_j=X_{j-1}$ for $j=2,\dots,k+1$. By definition, $(k+1)!$ times the left-hand side of (5.17) is equal to

$$\begin{aligned} &\sum_{\tau\in\mathfrak{S}_{k+1}}(-1)^\tau Z_{\tau(1)}\otimes\cdots\otimes Z_{\tau(k+1)}\\ &\quad=\sum_{\substack{1\leqslant j\leqslant k+1\\ \tau\in\mathfrak{S}_{k+1}:\ \tau(j)=1}}(-1)^\tau X_{\tau(1)-1}\otimes\cdots\otimes X_{\tau(j-1)-1}\otimes Y\otimes X_{\tau(j+1)-1}\otimes\cdots\otimes X_{\tau(k+1)-1}.\end{aligned}$$

For any $\tau\in\mathfrak{S}_{k+1}$ such that $\tau(j)=1$, let $\overline{\tau}\in\mathfrak{S}_k$ be the permutation determined by the commutativity of

$$\begin{array}{ccc} [1,k] & \longrightarrow & [1,k+1]\setminus\{j\}\\ \Big\downarrow{\scriptstyle\overline{\tau}} & & \Big\downarrow{\scriptstyle\tau}\\ [1,k] & \longrightarrow & [1,k+1]\setminus\{1\}. \end{array}$$

Then, $(-1)^\tau=(-1)^{\overline{\tau}}(-1)^{j-1}$ and the above is equal to

$$\sum_{\substack{1\leqslant j\leqslant k+1\\ \tau\in\mathfrak{S}_{k+1}:\ \tau(j)=1}}(-1)^{j-1}(-1)^{\overline{\tau}}X_{\overline{\tau}(1)}\otimes\cdots\otimes X_{\overline{\tau}(j-1)}\otimes Y\otimes X_{\overline{\tau}(j)}\otimes\cdots\otimes X_{\overline{\tau}(k)}$$

which proves (5.17). $\square$

Lemma 5.14. *For any $f\in\bigwedge^2\mathfrak{g}$ and $\eta\in\bigwedge^k\mathfrak{g}$, one has*

$$\sum_{1\leqslant a<b\leqslant k+1}(-1)^{a+b}\big[f^{ab},\eta^{a,1,\dots,\widehat{a},\dots,\widehat{b},\dots,k+1}+\eta^{b,1,\dots,\widehat{a},\dots,\widehat{b},\dots,k+1}\big]=-\frac{k+1}{2}[\![f,\eta]\!]. \tag{5.18}$$

Proof. We may assume that f, η are of the form

$$f = f_1 \wedge f_2 = \frac{1}{2}(f_1 \otimes f_2 - f_2 \otimes f_1),$$
$$\eta = \eta_1 \wedge \cdots \wedge \eta_k = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(k)}.$$

The left-hand side of (5.18) is then equal to

$$\begin{aligned}
&\frac{1}{2k!} \sum_{\substack{1 \leqslant a < b \leqslant k+1 \\ \sigma \in \mathfrak{S}_k}} (-1)^{a+b} (-1)^\sigma \\
&\quad \big[f_1^a f_2^b - f_2^a f_1^b, (\eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(k)})^{a,1,\ldots,\widehat{a},\ldots,\widehat{b},\ldots,k+1} \\
&\qquad + (\eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(k)})^{b,1,\ldots,\widehat{a},\ldots,\widehat{b},\ldots,k+1} \big] \\
&= \frac{1}{2k!} \sum_{\substack{1 \leqslant a < b \leqslant k+1 \\ \sigma \in \mathfrak{S}_k}} (-1)^{a+b} (-1)^\sigma \\
&\quad \eta_{\sigma(2)} \otimes \cdots \otimes \eta_{\sigma(a)} \otimes [f_1, \eta_{\sigma(1)}] \otimes \eta_{\sigma(a+1)} \otimes \cdots \\
&\qquad \otimes \eta_{\sigma(b-1)} \otimes f_2 \otimes \eta_{\sigma(b)} \otimes \cdots \otimes \eta_{\sigma(k)} \\
&\qquad - \eta_{\sigma(2)} \otimes \cdots \otimes \eta_{\sigma(a)} \otimes f_2 \otimes \eta_{\sigma(a+1)} \otimes \cdots \\
&\qquad \otimes \eta_{\sigma(b-1)} \otimes [f_1, \eta_{\sigma(1)}] \otimes \eta_{\sigma(b)} \otimes \cdots \otimes \eta_{\sigma(k)} \\
&\qquad - \eta_{\sigma(2)} \otimes \cdots \otimes \eta_{\sigma(a)} \otimes [f_2, \eta_{\sigma(1)}] \otimes \eta_{\sigma(a+1)} \otimes \cdots \\
&\qquad \otimes \eta_{\sigma(b-1)} \otimes f_1 \otimes \eta_{\sigma(b)} \otimes \cdots \otimes \eta_{\sigma(k)} \\
&\qquad + \eta_{\sigma(2)} \otimes \cdots \otimes \eta_{\sigma(a)} \otimes f_1 \otimes \eta_{\sigma(a+1)} \otimes \cdots \\
&\qquad \otimes \eta_{\sigma(b-1)} \otimes [f_2, \eta_{\sigma(1)}] \otimes \eta_{\sigma(b)} \otimes \cdots \otimes \eta_{\sigma(k)}.
\end{aligned} \tag{5.19}$$

Setting $\sigma' = \sigma \circ (1 \cdots a)$ in the first summand and $\sigma' = \sigma \circ (1 \cdots b-1)$ in the second, we see that their sum is equal to

$$\begin{aligned}
&\frac{1}{2k!} \sum_{\substack{1 \leqslant a < b \leqslant k+1 \\ \sigma \in \mathfrak{S}_k}} (-1)^{b-1} (-1)^\sigma \\
&\quad \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(a-1)} \otimes [f_1, \eta_{\sigma(a)}] \otimes \eta_{\sigma(a+1)} \otimes \cdots \\
&\qquad \otimes \eta_{\sigma(b-1)} \otimes f_2 \otimes \eta_{\sigma(b)} \otimes \cdots \otimes \eta_{\sigma(k)} \\
&- \frac{1}{2k!} \sum_{\substack{1 \leqslant a < b \leqslant k+1 \\ \sigma \in \mathfrak{S}_k}} (-1)^{a} (-1)^\sigma \\
&\quad \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(a-1)} \otimes f_2 \otimes \eta_{\sigma(a)} \otimes \cdots \\
&\qquad \otimes \eta_{\sigma(b-2)} \otimes [f_1, \eta_{\sigma(b-1)}] \otimes \eta_{\sigma(b)} \otimes \cdots \otimes \eta_{\sigma(k)}
\end{aligned}$$

and therefore to

$$\begin{aligned}
&\frac{1}{2k!}\sum_{\substack{1\leqslant a\neq b\leqslant k\\ \sigma\in\mathfrak{S}_k}}(-1)^{b-1}(-1)^\sigma\\
&\qquad\eta_{\sigma(1)}\otimes\cdots\otimes\eta_{\sigma(a-1)}\otimes[f_1,\eta_{\sigma(a)}]\otimes\eta_{\sigma(a+1)}\otimes\cdots\\
&\qquad\quad\otimes\eta_{\sigma(b-1)}\otimes f_2\otimes\eta_{\sigma(b)}\otimes\cdots\otimes\eta_{\sigma(k)}\\
&=\frac{k+1}{2}f_2\wedge\bigl(\mathrm{ad}(f_1)\eta\bigr)
\end{aligned}$$

where we used Lemma 5.13. Similarly, the sum of the last two summands in (5.19) is equal to

$$-\frac{k+1}{2}f_1\wedge\bigl(\mathrm{ad}(f_2)\eta\bigr).$$

Thus, the left-hand side of (5.18) is equal to

$$\frac{k+1}{2}\bigl(f_2\wedge\mathrm{ad}(f_1)\eta-f_1\wedge\mathrm{ad}(f_2)\eta\bigr)=-\frac{k+1}{2}[\![f_1\wedge f_2,\eta]\!]$$

as claimed. □

Proof of Proposition 5.10. Write

$$(\Phi)_F=\Phi_D+\hbar^{n+1}\xi+\hbar^{n+2}\psi\quad\mathrm{mod}\ \hbar^{n+3}$$

for some $\psi\in U\mathfrak{g}^{\otimes3}$. Since Φ_D is equal to 1 mod $\hbar^2$ and satisfies the pentagon equation with respect to $\Delta_F(\cdot)=F\Delta(\cdot)F^{-1}$, we have, mod $\hbar^{n+3}$,

$$\begin{aligned}
0=\mathrm{Pent}_{\Delta_F}\bigl((\Phi)_F\bigr)&=\mathrm{Pent}_{\Delta_F}(\Phi_D)+\hbar^{n+1}d_H^{\Delta_F}(\xi)+\hbar^{n+2}d_H^{\Delta_F}(\psi)\\
&=\hbar^{n+1}d_H^{\Delta_F}(\xi)+\hbar^{n+2}d_H\psi
\end{aligned}$$

where, for any $\eta\in U\mathfrak{g}^{\otimes3}$,

$$d_H^{\Delta_F}\eta=1\otimes\eta-\Delta_F\otimes\mathrm{id}^{\otimes2}(\eta)+\mathrm{id}\otimes\Delta_F\otimes\mathrm{id}(\eta)-\mathrm{id}^{\otimes2}\otimes\Delta_F(\eta)+\eta\otimes1$$

is equal to d_H mod $\hbar$. Applying Alt_4 to both sides, and using Lemmas 5.12 and 5.14, we find, with $\widetilde{\xi}=\mathrm{Alt}_3\,\xi\in(\bigwedge^3\mathfrak{g})^{\mathfrak{l}_D}$

$$\begin{aligned}
0&=\mathrm{Alt}_4\bigl(d_H^{\Delta_F}\xi\bigr)\\
&=\frac{\hbar}{2}\sum_{1\leqslant a<b\leqslant4}(-1)^{a+b}\bigl[f^{ab},\widetilde{\xi}^{ab,1,\dots,\widehat{a},\dots,\widehat{b},\dots,4}\bigr]\\
&=\frac{\hbar}{2}\sum_{1\leqslant a<b\leqslant4}(-1)^{a+b}\bigl[f^{ab},\widetilde{\xi}^{a,1,\dots,\widehat{a},\dots,\widehat{b},\dots,4}+\widetilde{\xi}^{b,1,\dots,\widehat{a},\dots,\widehat{b},\dots,4}\bigr]\\
&=-\hbar[\![f,\widetilde{\xi}]\!]
\end{aligned}$$

where we used the fact that, for any $x \in U\mathfrak{g}$

$$\Delta_F(x) = \Delta(x) + \hbar\big[f, \Delta(x)\big] \mod \hbar^2$$

so that

$$\Delta_F(x) - \Delta_F(x)^{21} = 2\hbar\big[f, \Delta(x)\big].$$

This proves our claim since

$$\overline{\pi}^3_D\widetilde{\xi} = \mathrm{Alt}_3\,\overline{\pi}^3_D\xi = 0. \qquad \square$$

6. Uniqueness of twists

Let

$$\Phi = 1 + \hbar^2\varphi + \cdots \in 1 + \hbar^2\big(U\mathfrak{g}^{\otimes 3}[[\hbar]]\big)^{\mathfrak{g}},$$
$$\Phi_D = 1 + \hbar^2\varphi_D + \cdots \in 1 + \hbar^2\big(U\mathfrak{g}_D^{\otimes 3}[[\hbar]]\big)^{\mathfrak{g}_D}$$

be two solutions of the pentagon equation (1.1) which are non-degenerate in the sense of Definition 5.1. Contrary to Section 5, we do not assume in this section that $\Phi_D = \overline{\pi}^3_D(\Phi)$ but merely that

$$\widetilde{\varphi}_D = \overline{\pi}^3_D(\widetilde{\varphi}) \tag{6.1}$$

where

$$\widetilde{\varphi} = \mathrm{Alt}_3\,\varphi \in \Big(\bigwedge^3 \mathfrak{g}\Big)^{\mathfrak{g}} \quad \text{and} \quad \widetilde{\varphi}_D = \mathrm{Alt}_3\,\varphi_D \in \Big(\bigwedge^3 \mathfrak{g}_D\Big)^{\mathfrak{g}_D}.$$

This implies in particular that if $(\cdot,\cdot)$, $(\cdot,\cdot)_D$ are the bilinear forms on $\mathfrak{g}$, $\mathfrak{g}_D$ corresponding to Φ, Φ_D via (5.2) respectively, then $(\cdot,\cdot)_D$ is the restriction of $(\cdot,\cdot)$ to $\mathfrak{g}_D$. We denote the corresponding standard solutions of the MCYBE for $\mathfrak{g}$ and $\mathfrak{g}_D$ by $r_{\mathfrak{g}}, r_{\mathfrak{g}_D}$. Let now

$$F_i = 1^{\otimes 2} + \hbar f_i + \cdots \in 1 + \hbar\big(U\mathfrak{g}^{\otimes 2}[[\hbar]]\big)^{\mathfrak{l}_D}, \quad i = 1, 2,$$

be two elements such that $(\Phi)_{F_i} = \Phi_D$. Since $d_H f_i = 0$, we have

$$\widetilde{f_i} = \mathrm{Alt}_2\, f_i \in \Big(\bigwedge^2 \mathfrak{g}\Big)^{\mathfrak{l}_D}.$$

Theorem 6.1. *Let F_1, F_2 be as above and assume that $\widetilde{f_i} = r_{\mathfrak{g}} - r_{\mathfrak{g}_D} \mod (\bigwedge^2 \mathfrak{l}_D)^{\mathfrak{l}_D} = \bigwedge^2 \mathfrak{c}_D$. Then,*

(i) *there exist elements*

$$u \in 1 + \hbar U\mathfrak{g}[[\hbar]]^{\mathfrak{l}_D} \quad \text{and} \quad \lambda \in \hbar \bigwedge^2 \mathfrak{c}_D[[\hbar]]$$

such that

$$F_2 = \exp(\lambda) \cdot u \otimes u \cdot F_1 \cdot \Delta(u)^{-1}. \tag{6.2}$$

(ii) *If*

$$\overline{\pi}^2_D(F_i) = 1^{\otimes 2}, \quad i = 1, 2, \tag{6.3}$$

u may be chosen such that $\overline{\pi}^1_D(u) = 1$.

(iii) *If*

$$F_i^{\Theta} = F_i^{21}, \quad i = 1, 2, \tag{6.4}$$

then $\lambda = 0$ *and u may be chosen such that* $u^{\Theta} = u$. *u is then unique with this property.*

(iv) *If* $|D_{\mathfrak{g}} \setminus D| \leqslant 1$, *then* $\lambda = 0$ *and u is unique up to multiplication by* $\exp(c)$ *for some* $c \in \hbar\mathfrak{c}_D[[\hbar]]$.

Proof. (i)–(ii). Set

$$f = r_{\mathfrak{g}} - r_{\mathfrak{g}_D}$$

and write

$$f_i = d_H g_i + f + \nu_i$$

for some $g_i \in U\mathfrak{g}^{\mathfrak{l}_D}$, where $\nu_i = \pi^2_D(\widetilde{f_i}) \in (\bigwedge^2 \mathfrak{l}_D)^{\mathfrak{l}_D} = \bigwedge^2 \mathfrak{c}_D$. Then, replacing F_i by

$$\exp(-\hbar\nu_i) \cdot (1 - \hbar g_i) \otimes (1 - \hbar g_i) \cdot F_i \cdot \Delta(1 - \hbar g_i)^{-1}$$

we may assume that

$$F_i = 1^{\otimes 2} + \hbar f \mod \hbar^2. \tag{6.5}$$

Note that if (6.3) holds, then, by Corollary 2.5

$$0 = \overline{\pi}^2_D f_i = d_H \overline{\pi}^1_D g_i$$

and, replacing g_i by $g_i - \overline{\pi}^2_D g_i$, we may assume that $\overline{\pi}^1_D(g_i) = 0$. Similarly, if (6.4) holds, then

$$d_H g_i^{\Theta} + f^{\Theta} + \nu_i^{\Theta} = d_H g_i + f^{21} + \nu_i^{21}.$$

Since $f^{\Theta} = -f = f^{21}$, this yields

$$\nu_i^{\Theta} = -\nu_i \quad \text{and} \quad d_H g_i^{\Theta} = d_H g_i$$

whence $\nu_i = 0$ since Θ acts as multiplication by $+1$ on $\bigwedge^2 \mathfrak{c}_D \subseteq \bigwedge^2 \mathfrak{h}$ and, replacing g_i by $1/2(g_i + g_i^{\Theta})$, we may assume that $g_i^{\Theta} = g_i$.

We wish now to construct two sequences

$$v_n \in U\mathfrak{g}^{\mathfrak{l}_D} \quad \text{and} \quad \mu_n \in \bigwedge^2 \mathfrak{c}_D$$

such that, setting

$$u_n = (1 + \hbar^n v_n) \cdots (1 + \hbar v_1) \quad \text{and} \quad \lambda_n = \hbar^n \mu_n + \cdots + \hbar \mu_1$$

one has

$$F_2 = \exp(\lambda_n) \cdot u_n \otimes u_n \cdot F_1 \cdot \Delta(u_n)^{-1} \tag{6.6}$$

mod $\hbar^{n+1}$. If (6.3) (respectively (6.4)) holds, we require in addition that $\overline{\pi}^1_D(v_n) = 0$ (respectively $v_n^\Theta = v_n$ and $\mu_n = 0$) for all n.

By (6.5), we may set $v_1 = 0 = \mu_1$. Assume therefore v_k, μ_k constructed for $k = 1, \ldots, n$ and some $n \geqslant 1$. Let F_1' be defined by the right-hand side of (6.6) so that

$$F_2 = F_1' + \hbar^{n+1} \eta \mod \hbar^{n+2} \tag{6.7}$$

for some $\eta \in (U\mathfrak{g}^{\otimes 2})^{\mathfrak{l}_D}$. One readily checks that $(\Phi)_{F_1'} = \Phi_D$. Subtracting from this equation $(\Phi)_{F_2} = \Phi_D$ and computing mod $\hbar^{n+2}$, we find that

$$d_H \eta = \eta^{23} + \mathrm{id} \otimes \Delta(\eta) - \Delta \otimes \mathrm{id}(\eta) - \eta^{12} = 0.$$

Moreover, $\overline{\pi}^2_D \eta = 0$ (respectively $\eta^\Theta = \eta^{21}$) if (6.3) (respectively (6.4)) holds. Thus, $\eta = d_H v + \mu$ for some $v \in U\mathfrak{g}^{\mathfrak{l}_D}$ and $\mu \in (\bigwedge^2 \mathfrak{g})^{\mathfrak{l}_D}$ such that

$$\overline{\pi}^1_D v = 0,$$
$$v^\Theta = v \quad \text{and} \quad \mu^\Theta = -\mu$$

if (6.3), (6.4) hold respectively. Set $v_{n+1} = -v$ and

$$\begin{aligned}
F_1'' &= \left(1 + \hbar^{n+1} v_{n+1}\right)^{\otimes 2} \cdot F_1' \cdot \Delta\left(1 + \hbar^{n+1} v_{n+1}\right)^{-1} \\
&= \left(1 + \hbar^{n+1} v_{n+1}\right)^{\otimes 2} \cdot \exp(\lambda_n) \cdot u_n^{\otimes 2} \cdot F_1 \cdot \Delta(u_n)^{-1} \cdot \Delta\left(1 + \hbar^{n+1} v_{n+1}\right)^{-1} \\
&= \exp(\lambda_n) \cdot \left(\left(1 + \hbar^{n+1} v_{n+1}\right) u_n\right)^{\otimes 2} \cdot F_1 \cdot \Delta\left(\left(1 + \hbar^{n+1} v_{n+1}\right) u_n\right)^{-1}
\end{aligned}$$

where the last equality stems from the fact that v_{n+1} is invariant under $\mathfrak{l}_D$. We have

$$F_2 = \exp\left(-\hbar^{n+1} \mu\right) F_1'' \mod \hbar^{n+2}$$

so the inductive step may be completed by setting $\mu_{n+1} = -\mu$ provided we can show that μ lies in $\bigwedge^2 \mathfrak{c}_D$. To see this, let

$$\overline{F}_2 = 1 + \hbar f + \hbar^2 f_2 + \cdots + \hbar^{n+1} f_{n+1}$$

be the truncation of F_2 mod $\hbar^{n+2}$ and define

$$\xi = \xi(f; f_2, \dots, f_{n+1}) \in \big(U\mathfrak{g}^{\otimes 3}\big)^{\mathfrak{l}_D}$$

by

$$1 \otimes \overline{F}_2 \cdot \mathrm{id} \otimes \Delta(\overline{F}_2) \cdot \Phi - \Phi_D \cdot \overline{F}_2 \otimes 1 \cdot \Delta \otimes \mathrm{id}(\overline{F}_2) = \hbar^{n+2}\xi \quad \operatorname{mod} \hbar^{n+3}.$$

By Lemma 5.4, $d_H\xi = 0$ and, by Corollary 5.6, $\mathrm{Alt}_3\,\xi = 0$ since $\overline{F}_2$ extends to a solution mod $\hbar^{n+3}$. Similarly, if $\overline{F}''_1$ is the truncation of F''_1 mod $\hbar^{n+2}$, the corresponding error ξ'' satisfies $d_H\xi'' = 0$ and $\mathrm{Alt}_3\,\xi'' = 0$. Since $\overline{F}''_1 = \overline{F}_2 + \hbar^{n+1}\mu$ mod $\hbar^{n+2}$ and, for $n \geqslant 1$

$$\begin{aligned}\xi'' - \xi &= \xi(f; f_2, \dots, f_{n+1} + \mu) - \xi(f; f_2, \dots, f_{n+1}) \\ &= f^{23}\big(\mu^{12} + \mu^{13}\big) + \mu^{23}\big(f^{12} + f^{13}\big) - f^{12}\big(\mu^{13} + \mu^{23}\big) - \mu^{12}\big(f^{13} + f^{23}\big)\end{aligned}$$

we find, using Lemma 5.7, that $[\![f, \mu]\!] = 0$. By Theorem 4.5, this implies that

$$\mu = [\![f, x]\!] + y$$

where $y \in \bigwedge^2 \mathfrak{c}_D$ and $x \in \mathfrak{g}^{\mathfrak{l}_D} = \mathfrak{c}_D \subseteq \mathfrak{h}$. Since f is of weight 0, $[\![f, x]\!] = -\operatorname{ad}(x) f = 0$ whence $\mu = y \in \bigwedge^2 \mathfrak{c}_D$.

(iii) Let $u \in 1 + \hbar(U\mathfrak{g}[\![\hbar]\!])^{\mathfrak{l}_D}$, with $u^{\Theta} = u$, be such that

$$u \otimes u \cdot F_1 \cdot \Delta(u)^{-1} = F_1. \tag{6.8}$$

We claim that $u = 1$. Assume that $u = 1$ mod $\hbar^n$ for some $n \geqslant 1$ and write $u = 1 + \hbar^n u_n$ mod $\hbar^{n+1}$, where $u_n \in U\mathfrak{g}^{\mathfrak{l}_D}$ is fixed by Θ. Taking the coefficient of $\hbar^{n+1}$ in (6.8) we find that $d_H u_n = 0$. This implies that u_n lies in $\mathfrak{g}$ and therefore in $\mathfrak{h}$ since it is of weight zero. Since Θ acts as -1 on $\mathfrak{h}$ however, $u_n = 0$ as claimed.

(iv) If $|D_{\mathfrak{g}} \setminus D| \leqslant 1$, then $\bigwedge^2 \mathfrak{c}_D = 0$ so that $\lambda = 0$. Let now $u \in 1 + \hbar(U\mathfrak{g}[\![\hbar]\!])^{\mathfrak{l}_D}$ be such that

$$u \otimes u \cdot F_1 \cdot \Delta(u)^{-1} = F_1 \tag{6.9}$$

and write $u = 1 + \hbar u_1$ mod $\hbar^2$. Taking the coefficient of $\hbar$ in (6.9), we find that $d_H u_1 = 0$ so that $u_1 \in \mathfrak{g}^{\mathfrak{l}_D} = \mathfrak{c}_D$. Now let $u^{(2)} = u \cdot \exp(-\hbar u_1) = 1 + \hbar^2 u_2$ mod $\hbar^2$. Repeating the above argument with $u^{(2)}$, we find that $u_2 \in \mathfrak{c}_D$ and finally that there exists a sequence $u_n \in \mathfrak{c}_D$, $n \geqslant 1$, such that

$$u = \prod_{n \geqslant 1} \exp\big(\hbar^n u_n\big) = \exp\Bigg(\sum_{n \geqslant 1} \hbar^n u_n\Bigg). \qquad \square$$

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